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NONLINEAR VIBRATIONS OF SIMPLY SUPPORTED, CIRCULAR CYLINDRICAL SHELLS, COUPLED TO QUIESCENT FLUID

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The nonlinear free and forced vibrations of a simply supported, circular cylindrical shell in contact with an incompressible and inviscid, quiescent and dense fluid are investigated. Donnell's shallow-shell theory is used, so that moderately large vibrations are analysed. The boundary conditions on radial displacement and the continuity of circumferential displacement are exactly satisfied, while axial constraint is satisfied on the average. The problem is reduced to a system of ordinary differential equations by means of the Galerkin method. The mode shape is expanded by using three degrees of freedom; in particular, two asymmetric modes (*driven* and *companion* modes), plus an axisymmetric mode are employed. The time dependence of each term of the expansion is general and the axisymmetric mode is obtained from a series involving all axisymmetric linear modes. Different tangential constraints can be imposed at the shell ends. Effects of both internal and external dense fluid are studied. Internally, the shell is considered completely filled, while externally, an unbounded fluid domain is considered around the shell in the radial direction. The solution is obtained both numerically and by the *Method of Normal Forms*. Numerical results are obtained for both free and forced vibrations of empty and water-filled shells.

1. INTRODUCTION

In some applications, the vibration response of cylindrical shells calculated by linear theory is inaccurate. When the vibration amplitude becomes comparable to the shell thickness, a nonlinear theory should be used.

Two main phenomena have been examined for large amplitude vibrations of cylindrical shells; they are (i) the response–frequency relationship in the vicinity of a resonant frequency and (ii) the occurrence of travelling wave response (Chen & Babcock 1975). In particular, it has been found that the resonant frequency is a function of the amplitude of vibration, generally displaying a softening behaviour.

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For a periodic excitation applied to the circular cylindrical shell, we expect a standing wave, symmetrical with respect to the point of application, in the case of linear vibrations. This symmetrical standing wave is the *driven mode*. For large amplitude vibrations, circumferentially travelling waves have been observed in the response of the shell. They can be described as the movement of the nodal lines of the driven mode. The travelling wave appears when a second standing wave (mode), the orientation of which is at $\pi/(2n)$ (where n is the number of nodal diameters) with respect to previous one, is added to the driven mode. This second mode is called the *companion mode*. It has the same modal shape and frequency of the driven mode. The presence of this second mode arises because of the axial symmetry of the shell and is due to nonlinear coupling.

Many studies have been performed on large amplitude vibrations of circular cylindrical shells and many different approaches to the problem have been tried. However, this research area is still far from being considered "well established". In this paper, initially a literature survey is made, as complete as possible, of the diverse approaches and theories which have been used, giving some of the distinctive or key results. Then, differences between these earlier studies and the present work are discussed.

1.1. Literature Review

This literature review is mainly focused on nonlinear (large amplitude) free and forced vibrations of closed circular cylindrical shells. The review is structured as follows. First, empty shells (shells *in vacuo*) are discussed, including also a few contributions on shells in a light medium; this is followed by shells in contact with dense fluid.

1.1.1. *Circular cylindrical shells* in vacuo

The first study on this subject is attributed to Reissner (1955), who isolated a single half-wave (lobe) of the vibration mode and analysed it for simply supported shells. By using Donnell's shallow-shell theory for thin-walled shells, he found that the nonlinearity could be either of the hardening or softening type, depending upon the geometry of the lobe. Donnell's shallow-shell theory introduces a stress function, in order to combine the three equations of equilibrium involving the shell displacements in the radial, circumferential and axial directions into two equations involving only the radial displacement and the stress function. Chu (1961) continued with Reissner's work, extending the analysis to closed cylindrical shells, as opposed to just curved panels. He found that nonlinearity in this case always leads to hardening type characteristics.

Nowinski (1963) confirmed Chu's results. However, Evensen (1963) proved that Nowinski's analysis is not so accurate, because it does not maintain zero transverse deflection at the ends of the shell. Furthermore, Evensen found that Reissner's and Chu's theories do not satisfy the continuity of in-plane circumferential displacement. Evensen (1967) also noted that experiments suggest that the nonlinearity should be of *softening* type, as observed by Olson (1965). Indeed, Olson (1965) observed a slight nonlinearity of the softening type in the experimental response of a thin seamless shell made of copper; the measured change in vibration frequency was less than 1% for a vibration amplitude equal to 2·5 times the shell thickness. To reconcile this most important discrepancy between theory and experiments, Evensen (1967) used the same method as previous investigators but with a different form of the assumed flexural mode shapes. He included the companion mode in the analysis. However, Evensen's assumed modes are not moment-free at the ends of the shell, as they should be for classical simply supported shells, and the homogeneous solution for the stress function is neglected. He studied the free vibrations and the response to a modal excitation without considering damping and discussed the stability of the response curves. Evensen (1968) extended his work to infinitely long shells vibrating in a mode in which the generators of the cylindrical surface remain straight and parallel, by using the method of harmonic balance, without considering the companion mode. He also wrote an interesting review paper (Evensen 1974).

It is important to note that, in most of the literature, the assumed mode shapes (those used by Evensen, for example) are derived following experimental observation that, in large amplitude vibrations, the shell does not spend equal time intervals deflected outwards and deflected inwards; also inwards maximum deflections, measured from equilibrium, are larger than outwards ones. Hence, the original idea for mode expansion was to add to asymmetric linear modes an axisymmetric term (mode) giving a contraction to the shell. The axisymmetric term added by Evensen (1967, 1968, 1974) is a square sine in the axial coordinate, the amplitude of which is related to amplitudes of linear terms in order to satisfy the continuity of the circumferential displacement.

Matsuzaki & Kobayashi studied theoretically simply supported circular cylindrical shells (Matsuzaki & Kobayashi 1969*a*) and then analysed theoretically and experimentally clamped circular cylindrical shells (Matsuzaki & Kobayashi 1969*b*). They used (Matsuzaki & Kobayashi 1969*a*) the same approach and mode expansion as Evensen (1967) (with the opposite sign to the axisymmetric term because they assumed positive deflection outwards). However, they discussed the effect of structural damping and studied also the travelling wave mode that was experimentally observed in large-amplitude vibrations of shells. Their analysis is based on Donnell's shallow-shell theory. In the paper addressed to clamped shells, Matsuzaki & Kobayashi (1969*b*) modified the mode expansion in order to satisfy the different boundary conditions and retained both the particular and the homogeneous solution for the stress function. Comparison with their own experimental results is also given. A softening type nonlinearity was found also in this case.

Dowell & Ventres (1968) used a different expansion† and approach in order to satisfy "on the average" some of the boundary conditions and to introduce tagential constraints. They obtained the particular and the homogeneous solutions for the stress function. Their interesting approach was followed by Atluri (1972), who found that some terms are missing in one of the equations used by Dowell & Ventres. The boundary conditions assumed by both Dowell & Ventres and by Atluri constrain the axial displacement at the shell extremities to be zero, so that they are different from the classical constraints of a simply supported shell (zero axial force at both ends). Atluri (1972) found that the nonlinearity is of the hardening type, in contrast to what was found in experiments. The axisymmetric term used by Dowell & Ventres and Atluri in their mode expansion is a sine in the axial coordinate (same shape of the first axisymmetric mode for linear vibrations). Their approach was criticized by Evensen (1978) because it gives a hardening result; he pointed out that satisfaction of continuity of circumferential displacement "on the average" is not a good enough approximation in view of its importance in nonlinear vibrations. Varadan *et al.* (1989) showed that hardening-type results based on the theory of Dowell & Ventres and Atluri are due to the choice of the axisymmetric term.

Leissa & Kadi (1971) studied linear and nonlinear vibrations of shallow shell segments, simply supported at the edges. This theory is not really suitable for the study of closed shells, but it is the first study on the effect of curvature of the generating lines on vibrations of shallow shells. El-Zaouk & Dym (1973) investigated free and forced vibrations of closed

†This mode expansion is an important innovation, and one that can be easily improved by adding additional axisymmetric terms.

shells having a curvature of the generating lines by using the Marguerre–von Karman–Donnell theory of shells, They also obtained numerical results for circular cylindrical shells; for this special case, their study is similar to Evensen's (1967), but effects of internal pressure and orthotropy are taken into account. Leissa (1973) also presents a literature review on large amplitude vibrations of circular cylindrical shells in a section of his very well known monograph *Vibration of Shells*.

Ginsberg used a different approach to solve the problem of asymmetric (Ginsberg 1973) and axisymmetric (Ginsberg 1974) vibrations of circular cylindrical shells. In fact, he avoided the assumptions of mode shapes and employed instead an asymptotic analysis to solve the nonlinear boundary conditions for a simply supported shell. The solution is reached by using the energy of the system to obtain the Lagrange equations, which are then solved via the harmonic balance method. Both softening or hardening nonlinearities were found, depending on some parameters of the system. Moreover, Ginsberg used the more accurate Flügge–Lur'e–Byrne shell theory, instead of Donnell's shallow-shell theory which was used in all the work discussed in the foregoing (excluding the two papers allowing curvature of the generating lines). Ginsberg (1973) studied the damped response to an excitation and its stability, considering both the driven and the companion modes.

Chen & Babcock (1975) used the perturbation method to solve the nonlinear equations obtained by Donnell's shallow-shell theory, without selecting a particular deflection solution. They solved the classical simply supported case and studied both the driven mode response, the companion mode participation, and the appearance of a travelling wave. A damped response to an external excitation was found. The solution involves a sophisticated mode expansion, including boundary layer terms in order to satisfy the boundary conditions. They also presented experimental results in good agreement with their theory, showing a softening nonlinearity. (It is noted that a few misprints are present in this paper.)

It is important to state that Ginsberg's (1973) numerical results and Chen & Babcock's (1975) numerical and experimental results constitute fundamental contributions to the study of the influence of the companion mode on the nonlinear forced response of circular cylindrical shells.

Radwan & Genin (1975, 1976) derived nonlinear modal equations by using the Sanders' nonlinear theory of shells and Lagrange equations of motion that they specialized to a thin cylinder simply supported at the ends. Numerical results give only the coefficients of the Duffing equation that they obtained by solving the problem; most of their results indicate a hardening-type nonlinearity (Radwan & Genin 1976). Kanaka Raju & Venkateswara Rao (1976) used the finite element method and employed Sanders' theory to study free vibrations of shells of revolution. They found hardening-type results for a closed circular cylindrical shell, in contradiction with all experiments available. Their paper was discussed by Evensen (1977), Prathap (1978), and a second time by Evensen (1978). In particular, Evensen (1977) commented that the authors ignored the physics of the problem, i.e., that thin shells bend more readily than they stretch. Nayfeh & Raouf (1987) studied the problem by using plane strain theory of shells and a perturbation analysis; thus, their study is more suitable for rings than for supported shells of finite length. They investigated the response when the frequency of the axisymmetric mode is approximately twice that of the asymmetric mode (two-to-one internal resonance).

Koval'chuck & Podchasov (1988) introduced a mode shape representation of the radial deflection, in order to allow the nodal lines to travel in the circumferential direction over time; it should be recognized that the same effect is also included in all the other theories that consider the companion mode. In fact, the same phenomenon was investigated by Matsuzaki & Kobayashi (1969*a*), for example, as previously discussed.

A mode shape expansion that can be considered a simple generalization of Evensen's (1967) was introduced by Varadan, Prathap & Ramani (1989); however, it is not moment-free at the ends of the cylinder. Donnell's shallow-shell theory was used and the results were compared with those obtained by using the mode expansion proposed by Dowell & Ventres (1968) and Atluri (1972). It was shown that the axisymmetric term of Dowell & Ventres and Atluri gives hardening-type results, as previously discussed. Tsai & Palazotto (1991) investigated cylindrical shell panels by using the finite element method. Adrianov & Kholod (1993) used an analytical solution for shallow cylindrical shells and plates based on Bolotin's asymptotic method, but they obtained results only for a rectangular plate. The generalized Berger's equations for shallow shells were employed.

Fu & Chia (1993) studied laminated shells by using the Timoshenko–Mindlin kinematic hypothesis, an extension of the Donnell theory of shells and a Fourier expansion of modes. Effects of transverse shear deformation, rotary inertia and geometrical initial imperfections are included in the analysis. The solution was obtained by the harmonic balance method after Galerkin projection. Softening or hardening nonlinearity was found, depending on the radius-to-thickness ratio. Only undamped free vibrations were investigated.

Chiba (1993*a*) studied experimentally large amplitude vibrations of two cantilevered circular cylindrical shells made of polyester sheet. He found that almost all responses have a softening nonlinearity. He observed that for modes with the same axial wavenumber, the weakest degree of softening nonlinearity can be attributed to the mode having the minimum natural frequency. He also found that shorter shells have a larger softening nonlinearity than taller ones. Travelling wave modes were also observed. Raouf & Palazotto (1994) studied the nonlinear free vibrations of curved orthotropic panels. They used Donnell's theory of shells and a combination of the Galerkin method and the Lindstedt–Poincaré perturbation method. Softening-type results were found. Kobayashi & Leissa (1994) studied free vibrations of shallow shells; they used the first-order shear deformation theory in order to study thick shells. Numerical results were obtained for spherical and paraboloid shells. Except for hyperbolic paraboloid shells, a softening behaviour was found for small vibration amplitudes.

Ganapathi & Varadan (1996) used the finite element method to study large-amplitude vibrations of circular cylindrical shells. They showed the participation of the axisymmetric contraction mode with the asymmetric linear modes, confirming the effectiveness of the mode expansions used by many authors, as discussed in the foregoing. Only free vibration were investigated in that paper, using Novozhilov's theory of shells. Ganapathi & Varadan also pointed out problems in the finite element analysis of closed shells that are not present in open shells. They extended their study to laminated composite circular cylindrical shells (Ganapathi & Varadan 1995). Selmane & Lakis (1997*a*) also applied the finite element method to study free vibrations of open and closed orthotropic cylindrical shells. Their method is a hybrid of the classical finite element method and shell theory. They used Sanders' theory of shells and obtained hardening nonlinearity for the same closed circular cylindrical shell simply supported at the ends investigated by Nowinski (1963) and Kanaka Raju & Venkateswara Rao (1976). However, the criticism raised by Evensen (1977) on Nowinski's and Kanaka Raju & Venkateswara Rao's results should be recalled here.

Some studies have addressed fluid–structure interaction in a light medium, like air, considering shell nonlinearity. For example, the nonlinear flutter of a circular cylindrical shell in supersonic flow was studied by Olson & Fung (1967) and Evensen & Olson (1967, 1968). In particular, Olson & Fung (1967) discussed the calculation of the aerodynamic pressure with the aid of both linear piston theory and potential flow theory; numerical results were obtained by using piston theory and a mode expansion involving two asymmetric modes having a different axial wavenumber plus two axisymmetric terms. Evensen & Olson (1967, 1968) used only piston theory, but discussed the mode expansion involving also companion modes.

1.1.2. *Circular cylindrical shells in contact with dense fluid*

All the foregoing studies dealt with shells vibrating in vacuum or in a light fluid medium such as air. Regarding nonlinear vibrations of cylindrical shells in contact with dense (heavy) fluids, the literature is less extensive. Chu & Kaña (1967) proposed a theory for the study of nonlinear vibrations of partially filled tanks; they employed a linear shell theory, and the nonlinearity is attributed to free surface waves.

The first study on vibrations of circular cylindrical shells in contact with a dense fluid considering shell nonlinearity can be attributed to Ramachandran (1979). He studied the large-amplitude vibrations of circular cylindrical shells having circumferentially varying thickness and immersed in a quiescent, inviscid and incompressible fluid, such that no free surface is present. Donnell's shell theory was used. Ramachandran used a mode expansion similar to that introduced by Evensen (1967), but with two important simplifications: (i) the companion mode was neglected; and (ii) the time dependence of both the asymmetric and the axisymmetric terms was assumed *a priori*. A solution was reached by the Rayleigh–Ritz method. Only free vibrations were investigated. Numerical results show a softening-type nonlinearity for shells *in vacuo* and for a shell submerged in water.

Boyarshina (1984, 1988) studied theoretically the nonlinear free and forced vibrations and stability of a circular cylindrical tank partially filled with a liquid and having a free surface. In this case, nonlinearity is attributed to the interaction of free surface waves and elastic flexural vibrations of the shell. In particular, three characteristic cases were investigated: (i) the natural frequency of sloshing of the free liquid surface equal to the natural frequency of shell vibration; (ii) twice the natural frequency of sloshing equal to the natural frequency of shell vibration; (iii) three times the natural frequency of sloshing equal to the natural frequency of shell vibration. Here "sloshing modes" are modes characteristic of the free liquid surface, where the amplitude of the free surface waves is larger than the wall displacement, in contrast to "bulging modes" that are characteristic of the oscillating tank wall exciting the liquid.

Gonçalves & Batista (1988) considered simply supported circular cylindrical shells filled with inviscid and incompressible fluid; actually, the constraints on axial displacement used approximate only the classical simple support. They used Sanders' nonlinear theory of shells and a novel mode expansion that includes two terms in the radial direction (the asymmetric and the axisymmetric ones) and ten terms to describe the in-plane displacements. The companion mode was not considered. Numerical results were obtained concerning the effect of the liquid on the nonlinear behaviour of shells. It was found that the presence of a dense fluid gives more strongly softening results *vis-à-vis* those for the same shell *in vacuo*. Both free vibrations and undamped response to harmonic excitation were investigated. No boundary conditions were imposed to the fluid at the ends of the shell.

Sivak & Telalov (1991) obtained experimental results on a vertical circular cylindrical shell made of titanium alloy in contact with water having a free surface. Experiments were performed with the shell partially filled and partially submerged in water. Experimental boundary conditions should simulate a clamped shell. Many experiments indicate softening nonlinearity; however, the completely filled shell shows a hardening nonlinearity, an effect increasing with the number of nodal diameters. It must be observed that the weight of liquid can introduce significant stresses in thin shells, and this effect can give a hardening behaviour in experiments. It seems more reasonable that water-filled shells not subject to gravity should show a significant softening nonlinearity.

Chiba (1993*b*) studied experimentally large-amplitude vibrations of two vertical cantilevered circular cylindrical shells made of polyester sheet partially filled with water to different levels. He observed that for bulging modes with the same axial wavenumber, the weakest degree of softening nonlinearity can be attributed to the mode having the minimum natural frequency, as observed for the same empty shells. He also found that shorter tanks have a larger softening nonlinearity than taller ones, as *in vacuo*. The tank with a lower liquid height has a greater softening nonlinearity than the tank with a higher liquid level. Travelling wave modes and coupling between two bulging modes (and between two sloshing modes) were also observed. Chiba (1993*b*) also investigated experimentally the effect of a thin film floating on the free surface and the behaviour of the free surface. Large-amplitude vibrations of two vertical clamped circular cylindrical shells, partially filled to different water levels were also studied by Chiba (1993*c*). In this case also, the responses display a generally softening nonlinearity. The shells tested show larger nonlinearity when partially filled, as compared to the empty and completely filled cases.

Selmane & Lakis (1997*b*) used a specialized finite element method to study the nonlinear dynamics of open cylindrical shells subjected to a flowing, inviscid and incompressible fluid. They employed Sanders' shell theory in their analysis. The companion mode, stability and forced response were not discussed. Both softening- and hardening-type nonlinearity was found for closed circular cylindrical shells, empty, filled and submerged in quiescent fluid, depending on the geometry. The influence of the nonlinearities associated with the fluid were found to be negligible on the dynamical behaviour of the shell–fluid system.

1.2. Present Study

In the present study, the nonlinear (large amplitude) free and forced vibrations of a simply supported, circular cylindrical shell in contact with an incompressible and inviscid, quiescent dense fluid are investigated. Donnell's nonlinear shallow-shell theory is used, so that moderately large vibrations can be analysed. The boundary conditions on radial displcement and continuity of circumferential displacement are exactly satisfied, while axial constraint is satisfied "on the average". It should be noted that the present approach synthesizes the characteristics of Evensen's (1967) solution (no radial deflection at the shell edges and exact continuity of circumferential displacement) and Dowell & Ventres' (1968) and Atluri's (1972) solutions (no radial deflection and no bending moment at the shell edges).

The problem is reduced to a system of ordinary differential equations by means of the Galerkin method, assuming an appropriate deflection shape. The mode shape is expanded by using two asymmetric modes (driven and companion modes) plus the axisymmetric mode, the importance of which in the nonlinear vibrations of circular cylindrical shells has been shown in several of the papers discussed in Section 1.1.1. The time dependence of each term of the expansion is general and the axisymmetric mode is obtained from a series involving all axisymmetric linear modes; in particular, the axisymmetric term has zero rotation and curvature at both the shell ends. Tangential constraints can be imposed to the shell, so that an infinite periodically supported shell is also studied. Only breathing modes, i.e., modes having a number of nodal diameters n larger than or equal to two, are considered. In fact, axisymmetric modes require a different mode shape expansion and are not very significant for engineering applications of finite length shells, while beam-bending modes (very interesting for pipes) have been studied extensively by Païdoussis and others; see, e.g., Païdoussis (1998). In the present study, both internal and external dense fluid is considered. The dynamic pressure in the fluid is assumed to be zero at the shell ends; no free liquid surface is present in this configuration. Internally, the shell is considered completely filled, while externally, an unbounded fluid domain is considered around the shell in the radial direction.

It appears that the present study is the first one allowing companion mode participation in the presence of interaction between the shell and a dense fluid.

The solution of the system of ordinary differential equations is obtained both numerically and by a perturbation method. The NDSolve routine of *Mathematica* software (Wolfram 1996) is used as the numerical integration algorithm. The perturbation method used here is the *Method of Normal Forms*, which is based upon a nonlinear transformation of variables that simplifies the system of equations, obtaining the so-called "normal form" of the system (Guckenheimer & Holmes, 1983; Wiggins 1990). This method is a valuable computational tool. It is suitable for the study of most local phenomena in nonlinear dynamics, such as internal resonances, the dynamics in the vicinity of a Hopf bifurcation, stationary and nonstationary motions. Moreover, this method permits a complete parametric study of the problem under investigation.

Numerical results are obtained for both free and forced vibrations of empty and water-filled shells. Results are compared to those obtained in some of the work discussed in Section 1.1. Frequency response plots in the case of companion mode participation are presented and discussed in the case of water-filled shells.

2. THEORETICAL FORMULATION

In this study, attention is focused on both a simply supported, closed circular cylindrical shell of length L and an infinitely long shell periodically supported. In the last case, the part of the shell considered lies between two supports, L apart, while the effect of the part of the shell external to this length is only considered as a constraint; only modes that are antisymmetric (in a shell-section along the axis) with respect to the support are considered in this case (lower-frequency modes). A cylindrical coordinate system (O; x, r, θ) is chosen, with the origin O placed at the centre of one end of the shell. The displacements of points in the middle surface of the shell are denoted by u, v, w in the axial, circumferential and radial direction, respectively (see Figure 1). Using Donnell's shallow-shell nonlinear theory, the equation of motion for large amplitude transverse vibrations of a circular cylindrical shell is given by (Evensen 1967; Chen & Babcock 1975)

$$D\nabla^4 w + ch\dot{w} + \rho h\ddot{w} = f + q + \frac{1}{R}\frac{\partial^2 F}{\partial x^2} + \left(\frac{\partial^2 F}{R^2\partial\theta^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{R\partial x\partial\theta}\frac{\partial^2 w}{R\partial x\partial\theta} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{R^2\partial\theta^2}\right), \quad (1)$$

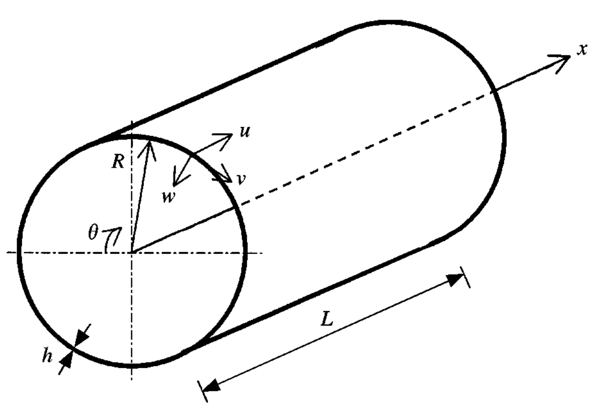

Figure 1. Shell geometry and coordinate system.

where $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity, E is Young's modulus, ν the Poisson ratio, h the shell thickness, R the mean shell radius, ρ the mass density of the shell, c (kg m^{-3} s^{-1}) the damping coefficient, and f and q are the radial pressures applied to the surface of the shell as a consequence of external forces and the fluid, respectively. The pressure $q = -q_I + q_E$ exerted by the fluid on the shell has two contributions, associated with the internal and external fluid, respectively. The radial deflection w is positive inward, $\dot{w} = (\partial w/\partial t)$, $\ddot{w} = (\partial^2 w/\partial t^2)$, and F is the in-plane stress function; F is given by the following relation (Evensen 1967; Chen & Babcock 1975):

$$\frac{1}{Eh}\nabla^4 F = -\frac{1}{R}\frac{\partial^2 w}{\partial x^2} + \left[\left(\frac{\partial^2 w}{R\partial x \partial\theta}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{R^2\partial\theta^2}\right]. \tag{2}$$

In equations (1) and (2) the biharmonic operator is defined as $\nabla^4 = [\partial^2/\partial x^2 + \partial^2/(R^2\partial\theta^2)]^2$. These equations are based on Donnell's shallow-shell theory, so that results are accurate only for modes of high circumferential number, n (n being the number of nodal diameters); in particular, $1/n^2 \ll 1$ must be satisfied, so that $n \geq 5$ is required in order to have fairly good accuracy. The forces per unit length in the axial and circumferential directions, as well as the shear force, are given by (Dowell & Ventres 1968; Atluri 1972)

$$N_x = \frac{1}{R^2}\frac{\partial^2 F}{\partial\theta^2}, \quad N_\theta = \frac{\partial^2 F}{\partial x^2}, \quad N_{x\theta} = -\frac{1}{R}\frac{\partial^2 F}{\partial x \partial\theta}. \tag{3}$$

Denoting by u and v the axial and the circumferential displacements, respectively, the strain–displacement relations are (Dowell & Ventres 1968; Atluri 1972)

$$(1-\nu^2)\frac{N_x}{Eh} = -\frac{\nu w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial^2 w}{R\partial\theta}\right)^2 + \frac{\partial u}{\partial x} + \frac{\nu}{R}\frac{\partial v}{\partial\theta}, \tag{4}$$

$$(1-\nu^2)\frac{N_\theta}{Eh} = -\frac{w}{R} + \frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{R\partial\theta}\right)^2 + \nu\frac{\partial u}{\partial x} + \frac{1}{R}\frac{\partial v}{\partial\theta}, \tag{5}$$

$$(1-\nu^2)\frac{N_{x\theta}}{Eh} = 2(1-\nu)\left[\frac{1}{R}\frac{\partial w}{\partial x}\frac{\partial w}{\partial\theta} + \frac{1}{R}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial x}\right]. \tag{6}$$

The flexural mode shapes w are expanded by using the linear modes as basis; in particular, the flexural response having n nodal diameters and m longitudinal half-waves can be written as follows:

$$w(x,\theta,t) = [A_{mn}(t)\cos(n\theta) + B_{mn}(t)\sin(n\theta)]\sin(\lambda_m x) + \sum_{m=1}^{\tilde{N}} A_{m0}(t)\sin(\lambda_m x), \tag{7a}$$

where $\lambda_m = m\pi/L$ and t is the time; $A_{mn}(t)$, $B_{mn}(t)$ and $A_{m0}(t)$ are unknown functions of time t. Equation (7a) was obtained by supposing that the nonlinear interaction among linear modes of the chosen basis involve only the asymmetric modes having the particular m and n values, and all the axisymmetric modes; therefore, the nonlinear interaction among asymmetric modes having different n and m values is neglected. In fact, axisymmetric modes play an important role in nonlinear vibrations, as pointed out in the Introduction. In the following analysis, the sum in equation (7a) is truncated at $\tilde{N} = 3$ in order to study only modes having one axial half-wave ($m = 1$). In particular, the following mode expansion is used:

$$w(x,\theta,t) = [A_{mn}(t)\cos(n\theta) + B_{mn}(t)\sin(n\theta)]\sin(\lambda_m x) + A_{m0}(t)[3\sin(\lambda_m x) - \sin(3\lambda_m x)]. \tag{7b}$$

Equation (7b) is obtained from equation (7a) by selecting only modes symmetrical with respect to the plane $x = L/2$ (for $m = 1$ we expect a symmetric response with respect to this plane), so that $A_{20}(t) = 0$, and imposing a further condition in order to reduce the system to three degrees of freedom; namely that there is zero contribution of the axisymmetric term to the rotation of the shell, $\partial w/\partial x$, at the ends. In fact, experiments show that inwards contraction in large amplitude vibrations is mainly located in the central part of the shell. It is interesting to note that the axisymmetric term of equation (7b) may be rewritten as $[3\sin(\lambda_m x) - \sin(3\lambda_m x)] = 4\sin^3(\lambda_m x)$. A larger $\tilde{N}$ value must be taken in equation (7a) to study modes involving $m > 1$; models having more than three degrees of freedom can also be studied for $m = 1$ and $m > 1$.

The mode expansion of equation (7b) may be considered as an improved version of that used by Dowell & Ventres (1968) and Atluri (1972). Limitations of the expansion used in these two papers have already been discussed in the Introduction. In the present study, the axisymmetric term of the expansion gives zero rotation at the shell ends and satisfies the moment-free boundary condition. Thus, the characteristics of Evensen's (1967) expansion (zero rotation of axisymmetric term) and Dowell & Ventres' (1968) (zero moment) were combined in the proposed expansion. Equation (7b) satisfies the boundary conditions

$$w = 0 \quad \text{and} \quad M_x = -D\{(\partial^2 w/\partial x^2) + \nu[\partial^2 w/(R^2 \partial\theta^2)]\} = 0 \quad \text{at } x = 0, L, \tag{8}$$

where M_x is the bending moment per unit length. The other boundary conditions differ for the simply supported shell of finite length (Case 1) and for the infinitely long, periodically supported shell with restrained axial displacement at the supports (Case 2). They are

$$\textit{Case 1:} \quad N_x = 0 \quad \text{at } x = 0, L \quad \text{and} \quad v = 0 \quad \text{at } x = 0, L, \tag{9a}$$

$$\textit{Case 2:} \quad u = 0 \quad \text{at } x = 0, L \quad \text{and} \quad v = 0 \quad \text{at } x = 0, L. \tag{9b}$$

Moreover, u, v and w must be continuous in θ. Case 1 corresponds to the classical simply supported shell. The conditions imposed in Case 2 are well justified by the reciprocal constraint between the part of the shell under consideration and extensions thereof outside $(0, L)$. Case 2 also approximates a shell with rings at the ends. By using the present formulation, it is also possible to study a shell subjected to axisymmetric prestress, i.e. $N_x = \tilde{N}_x$ and $N_\theta = \tilde{N}_\theta$ at $x = 0, L$.

3. FLUID–STRUCTURE INTERACTION

The nonlinear effects in the dynamic pressure and in the boundary conditions at the fluid–structure interface are neglected in this study. They were found to be negligible by Gonçalves & Batista (1988). The negligible effect of nonlinear terms due to fluid–structure interaction in the dynamics of coupled fluid–shell systems was also inferred by Ginsberg (1975) and Selmane & Lakis (1997*b*) in the case of quiescent fluids. Furthermore, the fluid is assumed to be incompressible and inviscid; these assumptions were shown to be in good agreement with experimental results for some liquids, such as water (see, e.g., Amabili 1996). The shell prestress due to the fluid weight is neglected. The fluid is either inside or outside the shell, or both; the fluid volume is limited between the shell ends, i.e., in the case of the external fluid, it extends radially to infinity, but only for $x \in [0, L]$. The fluid motion is described by the velocity potential Φ, which satisfies the Laplace equation

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0. \tag{10}$$

The velocity of the fluid is related to Φ by $\mathbf{v} = -\nabla\Phi$. For the boundary condition at the fluid–shell interface it is assumed that there is no cavitation; for the internal fluid it is given by

$$\left(\frac{\partial \Phi}{\partial r}\right)_{r=R} = \dot{w}. \tag{11}$$

Both ends of the fluid volume (in correspondence to the shell edges) are assumed to be open, so that a zero pressure is assumed there,

$$(\Phi)_{x=0} = (\Phi)_{x=L} = 0. \tag{12}$$

A solution of equation (10) satisfying condition (12) is given by

$$\Phi = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [a_{mn}(t)\cos(n\theta) + b_{mn}(t)\sin(n\theta)][c_{mn}\mathrm{I}_n(\lambda_m r) + d_{mn}\mathrm{K}_n(\lambda_m r)]\sin(\lambda_m x), \tag{13}$$

where $\mathrm{I}_n(r)$ and $\mathrm{K}_n(r)$ are the modified Bessel functions of the first and second kind, respectively, and order n. Equation (13) must satisfy boundary condition (11) and Φ must be finite (regular) at $r = 0$ (internal fluid) or at $r \to +\infty$ (external fluid). By using the assumed mode shapes w, given by equation (7b), the solution of the boundary value problem for internal fluid only is

$$\Phi = [\dot{A}_{mn}(t)\cos(n\theta) + \dot{B}_{mn}(t)\sin(n\theta)]\frac{\mathrm{I}_n(\lambda_m r)}{\lambda_m \mathrm{I}'_n(\lambda_m R)}\sin(\lambda_m x)$$
$$+ \dot{A}_{m0}(t)\left[\frac{3\mathrm{I}_0(\lambda_m r)}{\lambda_m \mathrm{I}'_0(\lambda_m R)}\sin(\lambda_m x) - \frac{\mathrm{I}_0(3\lambda_m r)}{3\lambda_m \mathrm{I}'_0(3\lambda_m R)}\sin(3\lambda_m x)\right], \tag{14}$$

where $\mathrm{I}'_n(r)$ is the derivative of $\mathrm{I}_n(r)$ with respect to its argument. Therefore, the pressure q_I exerted by the internal fluid on the shell is given by

$$q_I = \rho_{FI}(\dot{\Phi})_{r=R} = \rho_{FI}\left\{[\ddot{A}_{mn}(t)\cos(n\theta) + \ddot{B}_{mn}(t)\sin(n\theta)]\frac{\mathrm{I}_n(\lambda_m R)}{\lambda_m \mathrm{I}'_n(\lambda_m R)}\sin(\lambda_m x)\right.$$
$$\left. + \ddot{A}_{m0}(t)\left[\frac{3\mathrm{I}_0(\lambda_m R)}{\lambda_m \mathrm{I}'_0(\lambda_m R)}\sin(\lambda_m x) - \frac{\mathrm{I}_0(3\lambda_m R)}{3\lambda_m \mathrm{I}'_0(3\lambda_m R)}\sin(3\lambda_m x)\right]\right\}, \tag{15}$$

where ρ_{FI} is the mass density of the internal fluid. In the case of external fluid, the same expression for the pressure q is obtained; only $\mathrm{I}_n(r)$ is replaced by $\mathrm{K}_n(r)$. Thus, equations (10) and (12) are still correct, but $\lim_{r\to\infty}\Phi = 0$ instead of Φ being regular at $r = 0$. Therefore, the pressure exerted by the external fluid of density ρ_{FE} is given by

$$q_E = \rho_{FE}(\dot{\Phi})_{r=R} = \rho_{FE}\left\{[\ddot{A}_{mn}(t)\cos(n\theta) + \ddot{B}_{mn}(t)\sin(n\theta)]\frac{\mathrm{K}_n(\lambda_m R)}{\lambda_m \mathrm{K}'_n(\lambda_m R)}\sin(\lambda_m x)\right.$$
$$\left. + \ddot{A}_{m0}(t)\left[\frac{3\mathrm{K}_0(\lambda_m R)}{\lambda_m \mathrm{K}'_0(\lambda_m R)}\sin(\lambda_m x) - \frac{\mathrm{K}_0(3\lambda_m R)}{3\lambda_m \mathrm{K}'_0(3\lambda_m R)}\sin(3\lambda_m x)\right]\right\}. \tag{16}$$

Equations (15) and (16) show that the inertia effects due to the fluid are different for the asymmetric and the axisymmetric terms of the mode expansion. Hence, the fluid is expected to change the nonlinear behaviour of the fluid–shell system. Usually, the inertial effect of the fluid is larger for axisymmetric modes, so that the fluid should have an enhanced effect on the nonlinear behaviour of the shell.

4. REDUCTION TO A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Substituting the expansion of w, equation (7b), on the right-hand side of equation (2), the following partial differential equation for F is obtained:

$$\begin{aligned}\frac{1}{Eh}\nabla^4 F = &\frac{\lambda_m^2}{R}\{[A_{mn}(t)\cos(n\theta) + B_{mn}(t)\sin(n\theta)]\sin(\lambda_m x)\\ &+ 3A_{m0}(t)[\sin(\lambda_m x) - 3\sin(3\lambda_m x)]\}\\ &+ \frac{n^2\lambda_m^2}{4R^2}\{2[A_{mn}^2(t) + B_{mn}^2(t)]\cos(2\lambda_m x)\\ &+ 2[-A_{mn}^2(t) + B_{mn}^2(t)]\cos(2n\theta) - 4A_{mn}(t)B_{mn}(t)\sin(2n\theta)\\ &-2A_{m0}(t)A_{mn}(t)\cos(n\theta)[3 - 12\cos(2\lambda_m x) + 9\cos(4\lambda_m x)]\\ &- 2A_{m0}(t)B_{mn}(t)\sin(n\theta)[3 - 12\cos(2\lambda_m x) + 9\cos(4\lambda_m x)]\}. \end{aligned} \quad (17)$$

The expression for the stress function F can be written as

$$F = F_h + F_p, \quad (18)$$

where F_h is the homogeneous and F_p is the particular solution. The particular solution is given by

$$\begin{aligned}F_p = &c_1(t)\cos(n\theta) + c_2(t)\sin(n\theta) + c_3(t)\sin(\lambda_m x) + c_4(t)\cos(n\theta)\sin(\lambda_m x)\\ &+ c_5(t)\sin(n\theta)\sin(\lambda_m x) + c_6(t)\cos(2\lambda_m x) + c_7(t)\cos(2n\theta) + c_8(t)\sin(2n\theta)\\ &+ c_9(t)\cos(n\theta)\cos(2\lambda_m x) + c_{10}(t)\sin(n\theta)\cos(2\lambda_m x) + c_{11}(t)\sin(3\lambda_m x)\\ &+ c_{12}(t)\cos(n\theta)\cos(4\lambda_m x) + c_{13}(t)\sin(n\theta)\cos(4\lambda_m x),\end{aligned} \quad (19)$$

where the functions c_i, $i = 1, \dots, 13$, are given in Appendix A.

The expansion used for the transverse displacement w satisfies the boundary conditions given by equation (8). The boundary conditions for the in-plane displacements, equations (9), are satisfied "on the average". Specifically, the following conditions are imposed:

$$\int_0^{2\pi}\int_0^L N_x \,\mathrm{d}x\, R\,\mathrm{d}\theta = 0 \quad \text{(Case 1)}, \quad (20a)$$

$$\int_0^{2\pi}\int_0^L \frac{\partial u}{\partial x}\,\mathrm{d}x\, R\,\mathrm{d}\theta = \int_0^{2\pi}[u(L,\theta) - u(0,\theta)]R\,\mathrm{d}\theta = 0 \quad \text{(Case 2)}, \quad (20b)$$

and for both cases,

$$\int_0^{2\pi}\int_0^L \frac{\partial v}{R\partial\theta}\,\mathrm{d}x\, R\,\mathrm{d}\theta = \int_0^L [v(x,2\pi) - v(x,0)]\,\mathrm{d}x = 0, \quad (21)$$

$$\int_0^{2\pi}\int_0^L N_{x\theta}\,\mathrm{d}x\, R\,\mathrm{d}\theta = 0. \quad (22)$$

Equation (20a) assures a zero axial force N_x "on the average" and is applied for Case 1. Equation (20b) states that the axial displacement u is zero "on the average" at $x = 0, L$ and is applied for Case 2. Equation (21) assures that the circumferential displacement v is continuous "on the average" in θ; this condition is carefully examined in Appendix B, where it is proved that exact continuity of v is guaranteed by the present solution and not only continuity "on the average". Equation (22) is satisfied when u and w are continuous on the average in θ, and $v = 0$ on the average at $x = 0, L$.

The homogeneous solution of equation (17) can be assumed to be

$$F_h = \frac{1}{2}\bar{N}_x R^2\theta^2 + \frac{1}{2}x^2\left\{\bar{N}_\theta + \frac{1}{2\pi RL}\int_0^L\int_0^{2\pi}[\lambda_m^2 c_3(t)\sin(\lambda_m z) + 9c_{11}(t)\sin(3\lambda_m z)]\,R\,\mathrm{d}\theta\,\mathrm{d}z\right\}$$
$$-\bar{N}_{x\theta}xR\theta = \frac{1}{2}\bar{N}_x R^2\theta^2 + \frac{1}{2}x^2\left\{\bar{N}_\theta + \frac{1-(-1)^m}{L}\lambda_m[c_3(t) + 3c_{11}(t)]\right\} - \bar{N}_{x\theta}xR\theta, \tag{23}$$

where $\bar{N}_x$, $\bar{N}_\theta$ and $\bar{N}_{x\theta}$ are the in-plane restraint stresses generated at the ends of the shell, as a consequence of the in-plane constraints "on the average". Equation (23) is not the most general homogeneous solution, but it is chosen in order to satisfy the boundary conditions on the average. In fact, it satisfies equations (3) on the average, as a consequence of the contribution of F_p to $\bar{N}_\theta$ being $-1/(2\pi RL)\int_0^L\int_0^{2\pi}[\lambda_m^2 c_3(t)\sin(\lambda_m z) + 9c_{11}(t)\sin(3\lambda_m z)]\,R\,\mathrm{d}\theta\,\mathrm{d}z$ and the contributions of F_p to $\bar{N}_x$ and $\bar{N}_{x\theta}$ being zero. Boundary conditions (20)–(22) allow us to express the in-plane restraint stresses $\bar{N}_x$, $\bar{N}_\theta$ and $\bar{N}_{x\theta}$ [see equations (4)–(6)] in terms of w and its derivatives. In Case 1 they give

$$\bar{N}_x = 0, \tag{24a}$$

$$(1-\nu^2)\frac{\bar{N}_\theta}{Eh} = \frac{1}{2\pi RL}\int_0^{2\pi}\int_0^L\left[(\nu^2-1)\frac{w}{\mathrm{R}} + \frac{1-\nu^2}{2}\left(\frac{\partial w}{R\partial\theta}\right)^2\right]\mathrm{d}x\,R\,\mathrm{d}\theta, \tag{25a}$$

$$\bar{N}_{x\theta} = 0. \tag{26a}$$

In Case 2, they give

$$(1-\nu^2)\frac{\bar{N}_x}{Eh} = \frac{1}{2\pi RL}\int_0^{2\pi}\int_0^L\left[-\frac{\nu w}{\mathrm{R}} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{R\partial\theta}\right)^2\right]\mathrm{d}x\,R\,\mathrm{d}\theta, \tag{24b}$$

$$(1-\nu^2)\frac{\bar{N}_\theta}{Eh} = \frac{1}{2\pi RL}\int_0^{2\pi}\int_0^L\left[-\frac{w}{\mathrm{R}} + \frac{\nu}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{R\partial\theta}\right)^2\right]\mathrm{d}x\,R\,\mathrm{d}\theta, \tag{25b}$$

$$\bar{N}_{x\theta} = 0. \tag{26b}$$

In Case 1, simple calculations give

$$\bar{N}_\theta = Eh\left\{A_{m0}(t)\frac{8}{3\lambda_m LR}[(-1)^m - 1] + [A_{mn}^2(t) + B_{mn}^2(t)]\frac{n^2}{8R^2}\right\}; \tag{27}$$

in Case 2, the results are

$$\bar{N}_x = \frac{Eh}{1-\nu^2}\left\{A_{m0}(t)\frac{8\nu}{3\lambda_m LR}[(-1)^m - 1] + [A_{mn}^2(t) + B_{mn}^2(t)]\left(\frac{\lambda_m^2}{8} + \frac{\nu n^2}{8R^2}\right) + A_{m0}^2(t)\frac{9\lambda_m^2}{2}\right\}, \tag{28}$$

$$\bar{N}_\theta = \frac{Eh}{1-\nu^2}\left\{A_{m0}(t)\frac{8}{3\lambda_m LR}[(-1)^m - 1] + [A_{mn}^2(t) + B_{mn}^2(t)]\left(\frac{\nu\lambda_m^2}{8} + \frac{n^2}{8R^2}\right) + A_{m0}^2(t)\frac{\nu 9\lambda_m^2}{2}\right\}. \tag{29}$$

At this point, all the terms involved in the equation of motion, equation (1), have been evaluated.

By using the Galerkin method, three ordinary, coupled nonlinear differential equations can be obtained for the variables $A_{mn}(t)$, $B_{mn}(t)$ and $A_{m0}(t)$, by successively weighting the single original equation with suitable functions z_s, $s = 1, 2, 3$, and integrating over the shell middle surface. The weighting functions z_s are defined as

$$z_s(x,\theta) = \begin{cases} \cos(n\theta)\sin(\lambda_m x) & \text{for } s=1, \\ \sin(n\theta)\sin(\lambda_m x) & \text{for } s=2, \\ 3\sin(\lambda_m x) - \sin(3\lambda_m x) & \text{for } s=3. \end{cases} \tag{30}$$

The Galerkin projection, in this case, can be defined as

$$\langle D\nabla^4 w + ch\dot{w} + \rho h\ddot{w}, z_s\rangle = \int_0^{2\pi}\int_0^L (D\nabla^4 w + ch\dot{w} + \rho h\ddot{w}) z_s(x,\theta)\,\mathrm{d}x\,\mathrm{d}\theta. \tag{31}$$

The results of this operation are

$$\begin{aligned} \langle D\nabla^4 w + ch\dot{w} + \rho h\ddot{w}, z_s\rangle &= \frac{\pi L}{2}\left[A_{mn}(t)D\left(\lambda_m^2 + \frac{n^2}{R^2}\right)^2 + \dot{A}_{mn}(t)ch + \ddot{A}_{mn}(t)\rho h\right] \quad \text{for } s=1 \\ &= \frac{\pi L}{2}\left[B_{mn}(t)D\left(\lambda_m^2 + \frac{n^2}{R^2}\right)^2 + \dot{B}_{mn}(t)ch + \ddot{B}_{mn}(t)\rho h\right] \quad \text{for } s=2 \\ &= \pi L[A_{m0}(t)90D\lambda_m^4 + \dot{A}_{m0}(t)10ch + \ddot{A}_{m0}(t)10\rho h] \quad \text{for } s=3. \end{aligned} \tag{32}$$

The Galerkin projections for the fluid pressure, in the case of a fluid-filled shell, subjected to an external excitation $f = f_{mn}\cos(n\theta)\sin(\lambda_m x)\cos(\omega t)$ of unspecified physical origin give

$$\begin{aligned} \langle q_I, z_s\rangle &= -\frac{\pi L}{2}\rho_{FI}\frac{\mathrm{I}_n(\lambda_m R)}{\lambda_m \mathrm{I}_n'(\lambda_m R)}\ddot{A}_{mn}(t) + \frac{\pi L}{2} f_{mn}\cos(\omega t) \quad \text{for } s=1 \\ &= -\frac{\pi L}{2}\rho_{FI}\frac{\mathrm{I}_n(\lambda_m R)}{\lambda_m \mathrm{I}_n'(\lambda_m R)}\ddot{B}_{mn}(t) \quad \text{for } s=2 \\ &= -\pi L\rho_{FI}\left[\frac{9\mathrm{I}_0(\lambda_m R)}{\lambda_m \mathrm{I}_0'(\lambda_m R)} + \frac{\mathrm{I}_0(3\lambda_m R)}{3\lambda_m \mathrm{I}_0'(3\lambda_m R)}\right]\ddot{A}_{m0}(t) \quad \text{for } s=3. \end{aligned} \tag{33}$$

An analogous result is obtained for the effect of the external fluid. The projection of part of equation (1) involving the stress function F is quite tedious and was performed by using the *Mathematica* computer software (Wolfram 1996).

In conclusion, the following system of second-order ordinary differential equations was obtained for internal fluid:

$$\ddot{A}_{mn}(t) + 2\zeta_{mn}\omega_{mn}\dot{A}_{mn}(t) + \omega_{mn}^2 A_{mn}(t) + h_1 A_{mn}(t)A_{m0}(t) + h_2 A_{mn}^3(t) + h_2 A_{mn}(t)B_{mn}^2(t) + h_3 A_{mn}(t)A_{m0}^2(t) = \tilde{f}_{mn}\cos(\omega t), \tag{34}$$

$$\ddot{B}_{mn}(t) + 2\zeta_{mn}\omega_{mn}\dot{B}_{mn}(t) + \omega_{mn}^2 B_{mn}(t) + h_1 B_{mn}(t)A_{m0}(t) + h_2 B_{mn}^3(t) + h_2 B_{mn}(t)A_{mn}^2(t) + h_3 B_{mn}(t)A_{m0}^2(t) = 0, \tag{35}$$

$$\ddot{A}_{m0}(t) + 2\zeta_{m0}\omega_{m0}\dot{A}_{m0}(t) + \omega_{m0}^2 A_{m0}(t) + k_1 A_{mn}^2(t) + k_1 B_{mn}^2(t) + k_2 A_{m0}(t)A_{mn}^2(t) + k_2 A_{m0}(t)B_{mn}^2(t) + k_3 A_{m0}^2(t) + k_4 A_{m0}^3(t) = 0, \tag{36}$$

where h_i, $i = 1, 2, 3$, and k_i, $i = 1, \ldots, 4$, are coefficients depending on geometry, material properties and mode of the shell considered. In equations (34) and (35), ω_{mn} is the linear radian natural frequency of the fluid-shell system having m axial half-waves and n nodal diameters; ζ_{mn} is the corresponding damping ratio. The damping ratio is related to the damping coefficient c by $\zeta_{mn} = c/(2\omega_{mn}\rho_v)$, and the quantity $\tilde{f}_{mn}$ is related to the amplitude f_{mn} of the external pressure excitation by $\tilde{f}_{mn} = f_{mn}/(\rho_v h)$; here ρ_v is the "virtual density" of the fluid–shell system, given by $\rho_v = \rho + [\rho_F \mathrm{I}_n(\lambda_m R)]/[\lambda_m h \mathrm{I}_n'(\lambda_m R)]$ for internal fluid (and analogously for external fluid). In equation (36), ω_{m0} is not the linear natural frequency of the first axisymmetric mode, because we used a superposition of two modes in the axisymmetric term; this coefficient can be computed by Galerkin projection. The damping ratio for equation (36) is given by $\zeta_{m0} = c/(2\omega_{m0}\rho_{v0})$, where $\rho_{v0} = \rho + [\rho_F/(h\lambda_m)] \times \{9\mathrm{I}_0(\lambda_m R)/\mathrm{I}_0'(\lambda_m R) + \mathrm{I}_0(3\lambda_m R)/[3\mathrm{I}_0'(3\lambda_m R)]\}$ for the internal fluid.

As a consequence of the assumed external excitation appearing only in equation (34), it is possible to have a solution for $B_{mn}(t) = 0$ ($A_{mn}(t) \neq 0$, $A_{m0}(t) \neq 0$); this solution gives the so-called *driven mode*. The solution of $B_{mn}(t) \neq 0$ gives both the driven mode, related to $A_{mn}(t)$, and the *companion mode*, related to $B_{mn}(t)$. These solutions are investigated in Sections 5–7.

Equations (34)–(36) could easily be integrated using standard numerical schemes, such as a Runge–Kutta routine. On the other hand, a complete parametric study requires a very large number of integrations, including many initial conditions, variation of parameters and, where required, an accurate signal analysis. An alternative approach involves the use of approximate methods, such as perturbation methods, which allow a more systematic study of entire classes of solutions of the system of equations, with the approximation order considered.

5. PERTURBATION ANALYSIS

The perturbation method used in the present work is based on special nonlinear changes of variables. The aim of these transformations is to simplify the equation of motion as much as possible, in order to obtain a transformed system that could be solved analytically or more profitably integrated. The transformations are obtained by considering special equations called *homology equations*, which are solved in an approximate way. Moreover, the transformations should be invertible, leading to a complete approximate solution.

Let us first express the system (34)–(36) in nondimensional form by introducing the following nondimensional quantities:

$$\tau = \omega_{mn} t, \quad \tilde{q}_1(\tau) = A_{mn}(t)/h, \quad \tilde{q}_2(t) = B_{mn}(t)/h, \quad \tilde{q}_3(\tau) = A_{m0}(t)/h,$$

$$\tilde{h}_1 = \frac{h}{\omega_{mn}^2} h_1, \quad \tilde{h}_2 = \frac{h^2}{\omega_{mn}^2} h_2, \quad \tilde{h}_3 = \frac{h^2}{\omega_{mn}^2} h_3, \quad \tilde{k}_1 = \frac{h}{\omega_{mn}^2} k_1, \quad \tilde{k}_2 = \frac{h^2}{\omega_{mn}^2} k_2,$$

$$\tilde{k}_3 = \frac{h}{\omega_{mn}^2} k_3, \quad \tilde{k}_4 = \frac{h^2}{\omega_{mn}^2} k_4, \quad \omega_1 = \omega_2 = \omega_{mn}/\omega_{mn} = 1, \quad \omega_3 = \omega_{m0}/\omega_{mn}. \qquad (37)$$

In particular, ω_1, ω_2 and ω_3 are the nondimensional linear frequencies of $A_{mn}(t)$, $B_{mn}(t)$ and $A_{m0}(t)$, respectively; they have been obtained by dividing the linear frequencies by ω_{mn}. We analyse small nonlinear effects associated with moderately large amplitude of vibration. In order to separate the contribution of various terms present in equations (34)–(36), we introduce an ordering parameter ε as follows: $\tilde{q}_i = \varepsilon \hat{q}_i$, $i = 1, 2, 3$. The parameter ε will be set equal to 1 at the end of calculations, as suggested by Lichtenberg & Lieberman (1983). The modal equations (34)–(36) become

$$\ddot{\hat{q}}_i(\tau) + \varepsilon^2 \mu_i \dot{\hat{q}}_i(\tau) + \omega_i^2 \hat{q}_i(\tau) + \varepsilon \hat{f}_{2,i}(\mathbf{q}(\tau)) + \varepsilon^2 \hat{f}_{3,i}(\mathbf{q}(\tau)) = \varepsilon^2 \hat{r}_i(\tau), \quad i = 1, 2, 3, \tag{38}$$

where $\mathbf{q}(\tau) = [\hat{q}_1(\tau), \hat{q}_2(\tau), \hat{q}_3(\tau)]^{\mathrm{T}}$; $\hat{f}_{2,i}(\mathbf{q}(\tau))$, $i = 1, 2, 3$, contains the quadratic terms, and $\hat{f}_{3,i}(\mathbf{q}(\tau))$, $i = 1, 2, 3$, contains the cubic ones; also, $\Omega = \omega/\omega_{mn}$, $\hat{r}_1(\tau) = f_1 \cos(\Omega\tau)$, $\hat{r}_2(\tau) = 0$, $\hat{r}_3(\tau) = 0$, $f_1 = \varepsilon^{-3}\tilde{f}_{mn}/(h\omega_{mn}^2)$, $f_2 = f_3 = 0$, $\mu_1 = \mu_2 = 2\zeta_{mn}\varepsilon^{-2}$, $\mu_3 = 2\zeta_{m0}\varepsilon^{-2}\omega_{m0}/\omega_{mn}$, and ω is the circular frequency of the external load. The amplitudes of excitations f_2 and f_3 are zero as a consequence of the assumed external load directly exciting only the driven mode. Note that the external load gives an ε^2 effect in equation (38); in fact, this force amplitude is necessary in order to have a finite vibration amplitude under resonance conditions. In equation (38), damping also gives an ε^2 effect, in accordance with the assumption of small damping in the system.

Let us transform system (38) to first-order form with a diagonal linear part through the following change of variables:

$$\hat{q}_i = \xi_i + \xi_{i+N}, \quad \dot{\hat{q}}_i = \mathrm{j}\omega_i(\xi_i - \xi_{i+N}), \quad i = 1, \dots, N, \tag{39}$$

where $N = 3$, ξ_i is a complex variable, $\mathrm{j} = \sqrt{-1}$, $\xi_{i+N} = \bar{\xi}_i$, for $i = 1, \dots, N$, in which $\bar{\xi}_i$ is the complex conjugate of ξ_i. Moreover, we transform the previous nonautonomous system into autonomous form by introducing a new variable as follows:

$$\dot{z}_i = \mathrm{j}\Omega z_i, \text{ and then } z_i = -\frac{\mathrm{j}f_i}{4\omega_i}\exp(\mathrm{j}\Omega\tau), \quad i = 1, \dots, 2N, \tag{40}$$

where $z_{i+N} = \bar{z}_i$, and $f_{i+N} = f_i$, for $i = 1, \dots, N$ ($f_i = 0$, for $i \neq 1,4$). Therefore, we obtain the following complex autonomous system:

$$\begin{cases} \dot{\xi}_i = \mathrm{j}\omega_i\xi_i + \varepsilon f_{2,i}(\boldsymbol{\xi}) + \varepsilon^2 f_{3,i}(\boldsymbol{\xi}) - \varepsilon^2 \dfrac{\mu}{2}(\xi_i - \xi_{i+N}) + \varepsilon^2(z_i + z_{i+N}), \\ \dot{z}_i = \mathrm{j}\Omega z_i \end{cases} \quad i = 1, \dots, 2N, \tag{41}$$

where $\boldsymbol{\xi} = [\xi_1, \xi_2, \dots, \xi_{2N}]^{\mathrm{T}}$, $f_{2,i}(\boldsymbol{\xi}) = [\mathrm{j}/2\omega_i)]\hat{f}_{2,i}(\mathbf{q}(\boldsymbol{\xi}))$, $f_{3,i}(\boldsymbol{\xi}) = [\mathrm{j}/(2\omega_i)]\hat{f}_{3,i}(\mathbf{q}(\boldsymbol{\xi}))$. Note that in the previous $4N$-dimensional system only N equations are essential. In fact, the $2N$ equations involving ξ_i are complex conjugate, and the equations involving z_i are a mathematical artifice.

In the following part of Section 5, three different cases are studied by using perturbation analysis: (i) undamped free vibrations (Section 5.1); (ii) forced vibration with participation of the driven mode only (Section 5.2); and (iii) forced vibration with companion mode participation (Section 5.3). They are studied separately, because simplified approaches are applicable to cases (i) and (ii).

5.1. Free Vibrations of the Conservative System

Let us consider the special case where damping and external loads are not present; the governing modal equations become

$$\dot{\xi}_i = \mathrm{j}\omega_i\xi_i + \varepsilon f_{2,i}(\boldsymbol{\xi}) + \varepsilon^2 f_{3,i}(\boldsymbol{\xi}), \quad i = 1, 2, \dots, 2N, \tag{42}$$

where

$$f_{2,i}(\boldsymbol{\xi}) = \sum_{k,l=1}^{2N} f_{ikl}^{(2)}\xi_k\xi_l, \quad f_{3,i}(\boldsymbol{\xi}) = \sum_{k,l,m=1}^{2N} f_{iklm}^{(3)}\xi_k\xi_l\xi_m.$$

Let us consider the following nonlinear change of variables:

$$\xi_i = \eta_i + \varepsilon h_{2,i}(\boldsymbol{\eta}) + \varepsilon^2 h_{3,i}(\boldsymbol{\eta}) + \mathcal{O}(\varepsilon^3), \quad i = 1, \ldots, 2N, \tag{43}$$

where

$$\boldsymbol{\eta} = [\eta_1, \eta_2, \ldots, \eta_{2N}]^{\mathrm{T}}, \quad h_{2,i}(\boldsymbol{\eta}) = \sum_{k,l=1}^{2N} h_{ikl}^{(2)} \eta_k \eta_l, \quad h_{3,i}(\boldsymbol{\eta}) = \sum_{k,l,m=1}^{2N} h_{iklm}^{(3)} \eta_k \eta_l \eta_m.$$

The aim of the transformation is to simplify the dynamical system, cancelling most of the nonlinear terms. The cancellation can be complete in some special cases [hyperbolic systems; see the Hartmann–Grobmann linearization theorem in Guckenheimer & Holmes (1983)], when the transformed system is linear. For the general case, there is a part of the dynamical system, called *essential* (Guckenheimer & Holmes 1983) that cannot and should not be eliminated. For the problem under investigation, the transformed system assumes the following Normal Form (Brjuno 1971; Semler & Païdoussis 1996):

$$\dot{\eta}_i = \mathrm{j}\omega_i \eta_i + \varepsilon g_{2,i}(\boldsymbol{\eta}) + \varepsilon^2 g_{3,i}(\boldsymbol{\eta}) + \mathcal{O}(\varepsilon^3), \quad i = 1, \ldots, 2N, \tag{44}$$

where

$$g_{2,i}(\boldsymbol{\eta}) = \sum_{k,l=1}^{2N} g_{ikl}^{(2)} \eta_k \eta_l, \quad g_{3,i}(\boldsymbol{\eta}) = \sum_{k,l,m=1}^{2N} g_{iklm}^{(3)} \eta_k \eta_l \eta_m.$$

Note that the nonlinear transformation (43) and the Normal Form (44) have been expanded as power series of ε; this expansion is necessary to find an approximation of the transformation and the Normal Form itself. Substituting equation (43) into equation (42) and using equation (44) to eliminate the derivative with respect to time, we obtain

$$\begin{aligned} &\mathrm{j}\omega_i \eta_i + \varepsilon g_{2,i}(\boldsymbol{\eta}) + \varepsilon^2 g_{3,i}(\boldsymbol{\eta}) + \varepsilon \sum_{h=1}^{2N} \frac{\partial h_{2,i}(\boldsymbol{\eta})}{\partial \eta_h} (\mathrm{j}\omega_h \eta_h + \varepsilon g_{2,h}(\boldsymbol{\eta}) + \varepsilon^2 g_{3,h}(\boldsymbol{\eta}) + \cdots) \\ &\quad + \varepsilon^2 \sum_{h=1}^{2N} \frac{\partial h_{3,i}(\boldsymbol{\eta})}{\partial \eta_h} (\mathrm{j}\omega_h \eta_h + \varepsilon g_{2,h}(\boldsymbol{\eta}) + \varepsilon^2 g_{3,h}(\boldsymbol{\eta}) + \cdots) \\ &= \mathrm{j}\omega_i (\eta_i + \varepsilon h_{2,i}(\boldsymbol{\eta}) + \varepsilon^2 h_{3,i}(\boldsymbol{\eta}) + \cdots) + \varepsilon f_{2,i}(\boldsymbol{\eta}) + \varepsilon^2 \sum_{h=1}^{2N} \frac{\partial f_{2,i}(\boldsymbol{\eta})}{\partial \eta_h} h_{2,h}(\boldsymbol{\eta}) + \varepsilon^2 f_{3,i}(\boldsymbol{\eta}) + \cdots, \end{aligned} \tag{45}$$

where the omission points indicate the higher-order terms, of $\mathcal{O}(\varepsilon^3)$ and higher, in the Normal Form (44) and transformation (43); the function $f_{2,i}(\boldsymbol{\eta} + \varepsilon \mathbf{h}_2(\boldsymbol{\eta}) + \varepsilon^2 \mathbf{h}_3(\boldsymbol{\eta}) + \cdots)$ has been expanded to second order.

Let us now consider terms of the same order in ε separately:

$$\mathcal{O}(\varepsilon): \quad g_{2,i}(\boldsymbol{\eta}) + \sum_{h=1}^{2N} \frac{\partial h_{2,i}(\boldsymbol{\eta})}{\partial \eta_h} \mathrm{j}\omega_h \eta_h = \mathrm{j}\omega_i h_{2,i}(\boldsymbol{\eta}) + f_{2,i}(\boldsymbol{\eta}), \tag{46}$$

$$\begin{aligned} \mathcal{O}(\varepsilon^2): \quad & g_{3,i}(\boldsymbol{\eta}) + \sum_{h=1}^{2N} \frac{\partial h_{2,i}(\boldsymbol{\eta})}{\partial \eta_h} g_{2,h}(\boldsymbol{\eta}) + \sum_{h=1}^{2N} \frac{\partial h_{3,i}(\boldsymbol{\eta})}{\partial \eta_h} \mathrm{j}\omega_h \eta_h \\ &= \mathrm{j}\omega_i h_{3,i}(\boldsymbol{\eta}) + \sum_{h=1}^{2N} \frac{\partial f_{2,i}(\boldsymbol{\eta})}{\partial \eta_h} h_{2,h}(\boldsymbol{\eta}) + f_{3,i}(\boldsymbol{\eta}). \end{aligned} \tag{47}$$

Equations (46) and (47) are called *homology equations*. The solution of the first-order equations gives

$$\left.\begin{aligned} h_{ikl}^{(2)} &= \frac{f_{ikl}^{(2)}}{\mathrm{j}(\omega_k + \omega_l - \omega_i)} \\ g_{ikl}^{(2)} &= 0 \end{aligned}\right\} \quad \text{for } \omega_k + \omega_l - \omega_i \neq 0 \text{ and } i, k, l = 1, \ldots, 2N, \tag{48}$$

$$\left.\begin{aligned} h_{ikl}^{(2)} &= 0 \\ g_{ikl}^{(2)} &= f_{ikl}^{(2)} \end{aligned}\right\} \quad \text{for } \omega_k + \omega_l - \omega_i = 0 \text{ and } i, k, l = 1, \ldots, 2N. \tag{49}$$

The functions $g_{2,i}$ are called the *resonant* or *secular* part of $f_{2,i}$ and contain the essential part of the dynamical system. For studying the second-order effects, the second term on the left-hand side and the second term on the right-hand side of equation (47) must be expanded in explicit form:

$$\sum_{h=1}^{2N} \frac{\partial h_{2,i}(\boldsymbol{\eta})}{\partial \eta_h} g_{2,h}(\boldsymbol{\eta}) = \sum_{h=1}^{2N} \left[\frac{\partial}{\partial \eta_h} \left(\sum_{k,l=1}^{2N} h_{ikl}^{(2)} \eta_k \eta_l \right) \times \sum_{k,l=1}^{2N} g_{hkl}^{(2)} \eta_k \eta_l \right] = \sum_{k,l,m=1}^{2N} p_{iklm} \eta_k \eta_l \eta_m, \tag{50a}$$

$$\sum_{h=1}^{2N} \frac{\partial f_{2,i}(\boldsymbol{\eta})}{\partial \eta_h} h_{2,h}(\boldsymbol{\eta}) = \sum_{h=1}^{2N} \left[\frac{\partial}{\partial \eta_h} \left(\sum_{k,l=1}^{2N} f_{ikl}^{(2)} \eta_k \eta_l \right) \times \sum_{k,l=1}^{2N} h_{hkl}^{(2)} \eta_k \eta_l \right] = \sum_{k,l,m=1}^{2N} \hat{p}_{iklm} \eta_k \eta_l \eta_m, \tag{50b}$$

where $p_{iklm} = \sum_{h=1}^{2N} (h_{ihm}^{(2)} + h_{imh}^{(2)}) g_{hkl}^{(2)}$ and $\hat{p}_{iklm} = \sum_{h=1}^{2N} (f_{ihm}^{(2)} + f_{imh}^{(2)}) h_{hkl}^{(2)}$. As a consequence of these definitions, equation (47) becomes

$$\sum_{k,l,m=1}^{2N} [g_{iklm}^{(3)} + p_{iklm} + \mathrm{j}(\omega_k + \omega_l + \omega_m) h_{iklm}^{(3)}] \eta_k \eta_l \eta_m = \sum_{k,l,m=1}^{2N} [\mathrm{j}\omega_i h_{iklm}^{(3)} + \hat{p}_{iklm} + f_{iklm}^{(3)}] \eta_k \eta_l \eta_m. \tag{51}$$

Collecting terms of the same order in $\eta_k \eta_l \eta_m$ we obtain

$$\left.\begin{aligned} h_{iklm}^{(3)} &= \frac{\hat{p}_{iklm} + f_{iklm}^{(3)} - p_{iklm}}{\mathrm{j}(\omega_k + \omega_l + \omega_m - \omega_i)} \\ g_{iklm}^{(3)} &= 0 \end{aligned}\right\} \quad \text{for } \omega_k + \omega_l + \omega_m - \omega_i \neq 0 \text{ and } i, k, l, m = 1, \ldots, 2N, \tag{52a}$$

$$\left.\begin{aligned} h_{iklm}^{(3)} &= 0 \\ g_{iklm}^{(3)} &= f_{iklm}^{(3)} \end{aligned}\right\} \quad \text{for } \omega_k + \omega_l + \omega_m - \omega_i = 0 \text{ and } i, k, l, m = 1, \ldots, 2N. \tag{52b}$$

For the problem under investigation, if only one-to-one internal resonance between the driven and the companion modes is present (i.e., no other internal and combination resonances are present), we obtain the following expression of the Normal Form:

$$\begin{aligned} \dot{\eta}_1 &= \mathrm{j}\omega_1 \eta_1 + \varepsilon^2 (g_{1,1,2,5}^{(3)} \eta_1 \eta_2 \eta_5 + g_{1,1,3,6}^{(3)} \eta_1 \eta_3 \eta_6 + g_{1,1,1,4}^{(3)} \eta_1^2 \eta_4 + g_{1,2,2,4}^{(3)} \eta_2^2 \eta_4), \\ \dot{\eta}_2 &= \mathrm{j}\omega_2 \eta_2 + \varepsilon^2 (g_{2,1,2,4}^{(3)} \eta_1 \eta_2 \eta_4 + g_{2,2,3,6}^{(3)} \eta_2 \eta_3 \eta_6 + g_{2,1,1,5}^{(3)} \eta_1^2 \eta_5 + g_{2,2,2,5}^{(3)} \eta_2^2 \eta_5), \\ \dot{\eta}_3 &= \mathrm{j}\omega_3 \eta_3 + \varepsilon^2 (g_{3,2,3,5}^{(3)} \eta_2 \eta_3 \eta_5 + g_{3,1,3,4}^{(3)} \eta_1 \eta_3 \eta_4 + g_{3,3,3,6}^{(3)} \eta_3^2 \eta_6), \end{aligned} \tag{54}$$

where the nonlinear part of equation (54) is given in Appendix C.

For suitable initial conditions (i.e., $\eta_i(t) = 0$, $i = 2, 3$), the system can be reduced to the form

$$\dot{\eta}_1 = \mathrm{j}\omega_1 \eta_1 + \varepsilon^2 g_{1,1,1,4}^{(3)} \eta_1^2 \eta_4. \tag{55}$$

Equation (55) can easily be solved by introducing the polar form $\eta_1(\tau) = a_1 \exp[\mathrm{j}(\omega_1\tau + \vartheta_1(\tau))]$; separating the real and imaginary parts, equation (55) becomes

$$\dot{a}_1 = 0, \qquad \dot{\vartheta}_1 = \varepsilon^2 a_1^2 \mathscr{I}m(g^{(3)}_{1,1,1,4}). \tag{56}$$

This implies that the nondimensional free vibration frequency is given by $\omega_1 + \varepsilon^2 a_1^2 \mathscr{I}m(g^{(3)}_{1,1,1,4})$, and the physical free vibration is mainly governed by the following relationship:

$$A_{mn}(t) = \tilde{A}_{mn} \cos\left[\omega_{mn}\left(1 + \frac{\tilde{A}^2_{mn}}{4h^2}\mathscr{I}m(g^{(3)}_{1,1,1,4})\right)t\right], \tag{57}$$

where $\tilde{A}_{mn} = 2\varepsilon h a_1$ is the dimensional amplitude of vibration.

5.2. Forced Vibration: Driven Mode Only

In the presence of an external load and damping, the procedure is quite similar to the previous one, with the main difference that the phase space must be enlarged to include the variable z_1 and its complex conjugate; the presence of this new variable will affect both the nonlinear transformation and the Normal Form.

If the external excitation frequency is close to ω_{mn}, we can substitute the linear nondimensional frequency with $\omega_1 = \Omega + \varepsilon^2\sigma$, where σ is called *detuning* parameter. With the previous assumptions, the Normal Form becomes

$$\dot{\eta}_1 = \mathrm{j}\Omega\eta_1 + \varepsilon^2\left(g_{3,1}(\boldsymbol{\eta}) + \mathrm{j}\sigma\eta_1 - \frac{\mu_1}{2}\eta_1 + z_1\right),$$

$$\dot{\eta}_2 = \mathrm{j}\omega_2\eta_2 + \varepsilon^2\left(g_{3,2}(\boldsymbol{\eta}) - \frac{\mu_2}{2}\eta_2\right),$$

$$\dot{\eta}_3 = \mathrm{j}\omega_3\eta_3 + \varepsilon^2\left(g_{3,3}(\boldsymbol{\eta}) - \frac{\mu_3}{2}\eta_3\right), \tag{58}$$

where the nonlinear functions $g_{3,i}(\boldsymbol{\eta})$ are the same as in equation (54) and the $\mathcal{O}(\varepsilon^2)$ transformation $\mathbf{h}_3 = \mathbf{h}_3(\boldsymbol{\eta}, \mathbf{z})$ depends also on the vector $\mathbf{z} = [z_1, z_4]^{\mathrm{T}}$.

For suitable initial conditions (i.e., $\eta_i(t) = 0$, $i = 2, 3$), the system can be reduced to the form

$$\dot{\eta}_1 = \mathrm{j}\Omega\eta_1 + \varepsilon^2\left(g^{(3)}_{1,1,1,4}\eta_1^2\eta_4 + \mathrm{j}\sigma\eta_1 - \frac{\mu_1}{2}\eta_1 - \frac{\mathrm{j}f_1}{4\omega_1}\exp(\mathrm{j}\Omega\tau)\right). \tag{59}$$

Introducing the polar form $\eta_1(\tau) = a_1 \exp[\mathrm{j}(\Omega\tau + \vartheta_1(\tau))]$ and separating the real and the imaginary parts of equation (59), we obtain

$$\dot{a}_1 = \varepsilon^2\left(-\frac{\mu_1}{2}a_1 - \frac{f_1}{4\omega_1}\sin(\vartheta_1)\right), \qquad \dot{\vartheta}_1 a_1 = \varepsilon^2\left(\mathscr{I}m(g^{(3)}_{1,1,1,4})a_1^3 + \sigma a_1 - \frac{f_1}{4\omega_1}\cos(\vartheta_1)\right). \tag{60}$$

Searching for the fixed points of the previous system (i.e., $\dot{a}_1 = 0$, $\dot{\vartheta}_1 = 0$), we obtain a system of two nonlinear equations, that can be solved for the detuning parameter σ as function of the amplitude (Nayfeh & Mook 1979):

$$\sigma = -\mathscr{I}m(g^{(3)}_{1,1,1,4})a_1^2 \pm \sqrt{\frac{f_1^2}{16\omega_1^2 a_1^2} - \frac{\mu_1^2}{4}}. \tag{61}$$

Recalling that the amplitude of the load is $\tilde{f}_{mn} = \varepsilon^3 h \omega_{mn}^2 f_1$, and the linear damping $2\zeta_{mn} = \varepsilon^2 \mu_1$, we obtain

$$\Omega = 1 - \left(-\mathscr{Im}(g_{1,1,1,4}^{(3)}) \frac{\tilde{A}_{mn}^2}{4h^2} \pm \sqrt{\frac{\tilde{f}_{mn}^2}{4\tilde{A}_{mn}^2 \omega_{mn}^4} - \zeta_{mn}^2} \right). \tag{62}$$

5.3. Forced Vibration: Driven and Companion Modes

We consider in this section the previous case when the condition $\eta_2(t) = 0$ is eliminated [condition $\eta_3(t) = 0$ is retained]. In this case it is convenient to substitute both the linear natural frequencies $\omega_1 = \omega_2 = \Omega + \varepsilon^2 \sigma$ and to introduce the polar form $\eta_1(\tau) = a_1 \exp[\mathrm{j}(\Omega\tau + \vartheta_1(\tau))]$, $\eta_2(\tau) = a_2 \exp[\mathrm{j}(\Omega\tau + \vartheta_2(\tau))]$. Using these definitions, one obtains the following four-dimensional dynamical system:

$$\begin{aligned}
\dot{a}_1 &= \varepsilon^2 \left(\mathscr{Im}(g_{1,2,2,4}^{(3)}) a_1 a_2^2 \sin(2\vartheta_1 - 2\vartheta_2) - \frac{\mu_1}{2} a_1 - \frac{f_1}{4\omega_1} \sin \vartheta_1 \right), \\
\dot{\vartheta}_1 &= \varepsilon^2 \left(\mathscr{Im}(g_{1,1,2,5}^{(3)}) a_2^2 + \mathscr{Im}(g_{1,1,1,4}^{(3)}) a_1^2 + \mathscr{Im}(g_{1,2,2,4}^{(3)}) a_2^2 \cos(2\vartheta_1 - 2\vartheta_2) \right. \\
&\quad \left. + \sigma - \frac{f_1}{4\omega_1 a_1} \cos \vartheta_1 \right), \\
\dot{a}_2 &= \varepsilon^2 a_2 \left(-\mathscr{Im}(g_{2,1,1,5}^{(3)}) a_1^2 \sin(2\vartheta_1 - 2\vartheta_2) - \frac{\mu_1}{2} \right), \\
\dot{\vartheta}_2 &= \varepsilon^2 \left(\mathscr{Im}(g_{2,1,2,4}^{(3)}) a_1^2 + \mathscr{Im}(g_{2,1,1,5}^{(3)}) a_1^2 \cos(2\vartheta_1 - 2\vartheta_2) + \mathscr{Im}(g_{2,2,2,5}^{(3)}) a_2^2 + \sigma \right).
\end{aligned} \tag{63}$$

Stationary solutions of equation (38) are obtained by finding the fixed points of equations (63). The fixed points of such a system cannot be found analytically; therefore, a numerical procedure, must be used. In order to find numerical results, we used the damped Newton–Raphson method implemented in the *Mathematica* software (Wolfram 1996) varying the initial conditions, until an appropriate set was found. The difficulty is to find a point on the response curve; then it is easier to follow the solution for different frequencies of excitation.

5.4. Stability of the Solution

System (63) can be written in the following vector form:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)), \tag{64}$$

where $\mathbf{x}(t) = \{a_1(t), \vartheta_1(t), a_2(t), \vartheta_2(t)\}^{\mathrm{T}}$. Fixed points of the system are determined by

$$\mathbf{f}(\mathbf{x}(t)) = \mathbf{0}, \tag{65}$$

and they are denoted by $\bar{\mathbf{x}}$. The following change of coordinates is introduced:

$$\mathbf{x}(t) = \hat{\mathbf{x}}(t) + \bar{\mathbf{x}}, \tag{66}$$

where $\hat{\mathbf{x}}(t)$ is the new variable, measured from the fixed point. Using the Taylor expansion of the vectorial function $\mathbf{f}$, we can write

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t) + \bar{\mathbf{x}}) = \left(\frac{\partial \mathbf{f}(\hat{\mathbf{x}}(t) + \bar{\mathbf{x}})}{\partial \hat{\mathbf{x}}} \right)_{\hat{\mathbf{x}} = \mathbf{0}} \hat{\mathbf{x}} + \cdots. \tag{67}$$

Equation (67) shows that stability of the solution can be checked by computing the eigenvalues of the Jacobian matrix of the function $\mathbf{f}$ calculated in $\hat{\mathbf{x}} = \mathbf{0}$. Stable solutions are associated with negative real part in the four eigenvalues.

6. TRAVELLING WAVE MODE

The presence of the companion mode in the response of the shell leads to the appearance of a travelling wave. The flexural mode shapes are represented by equation (7b). Supposing that $A_{mn}(t) = \tilde{A}_{mn}\cos(\omega t + \vartheta_1)$ and $B_{mn}(t) = \tilde{B}_{mn}\cos(\omega t + \vartheta_2)$, equation (7b) can be rearranged as

$$w = \{[\tilde{A}_{mn}\cos(\omega t + \vartheta_1) + \tilde{B}_{mn}\sin(\omega t + \vartheta_2)]\cos(n\theta) - \tilde{B}_{mn}\sin(n\theta - \omega t - \vartheta_2)\}\sin(\lambda_m x) + \mathcal{O}(\varepsilon^2), \tag{68}$$

where ϑ_1 and ϑ_2 are the phase constants solution of equations (63). Equation (68) gives a travelling wave of amplitude $\tilde{B}_{mn}$ and radian frequency $\omega_T = \omega/n$, and a standing wave.

In general, the existence of driven and companion modes leads to the appearance of a travelling wave and a standing wave. As a consequence of both these modes appearing for forced nonlinear vibrations, this phenomenon represents a fundamental difference *vis-à-vis* linear vibrations.

7. LIMITING CASES

Now let us consider the two limits as $L/R \to 0$ and ∞ for shells in vacuum. Initially, the case $L/R \to 0$ is considered for $u = 0$ at $x = 0, L$ [equation (9b)]; it corresponds to a flat plate of dimension L in one direction and infinite in the orthogonal direction, simply supported at the edges (zero deflection and bending moment), with $u = 0$ on the edges normal to u, and $v = 0$ on the edges normal to v. If we divide the equations by $R^2\lambda_m^2$, only the functions c_1, c_2, c_7, c_8 are different from zero in the particular solution F_p [equation (19)]. In this case only N_x gives a contribution to the homogeneous solution, and it is simplified into

$$\bar{N}_x = \frac{Eh}{1 - \nu^2}\left\{[A_{mn}^2(t) + B_{mn}^2(t)]\frac{\lambda_m^2}{8} + A_{m0}^2(t)\frac{9\lambda_m^2}{2}\right\}.$$

The three equations of motion are

$$\ddot{A}_{mn}(t)\rho h + \dot{A}_{mn}(t)ch + A_{mn}(t)D\lambda_m^4 = \frac{-Eh\lambda_m^3 m\pi}{16(1 - \nu^2)L}A_{mn}(t)\{[A_{mn}^2(t) + B_{mn}^2(t)](3 - \nu^2) + 72A_{m0}^2(t)(2 - \nu^2)\} + f_{mn}\cos(\omega t), \tag{69}$$

$$\ddot{B}_{mn}(t)\rho h + \dot{B}_{mn}(t)ch + B_{mn}(t)D\lambda_m^4 = \frac{-Eh\lambda_m^3 m\pi}{16(1 - \nu^2)L}B_{mn}(t)\{[A_{mn}^2(t) + B_{mn}^2(t)](3 - \nu^2) + 72A_{m0}^2(t)(2 - \nu^2)\}, \tag{70}$$

$$\ddot{A}_{m0}(t)10\rho h + \dot{A}_{m0}(t)10ch + A_{m0}(t)90D\lambda_m^4 = \frac{-9Eh\lambda_m^3 m\pi}{4(1 - \nu^2)L}A_{m0}(t) \times\{[A_{mn}^2(t) + B_{mn}^2(t)](2 - \nu^2) + 36A_{m0}^2(t)\}. \tag{71}$$

The structure of equations (69)–(71) allows separating the problem into three independent equations by choosing the following initial conditions: $A_{mn}(t) \neq 0$ and $B_{mn}(t) = A_{m0}(t) = 0$. In particular, from equation (69) for $m = 1$, the following result is obtained:

$$\ddot{A}_{mn}(t)\rho h + \dot{A}_{mn}(t)ch + A_{mn}(t)\frac{Eh^3\pi^4}{12(1-\nu^2)L^4} = \frac{-Eh\pi^4}{16L^4}\frac{(3-\nu^2)}{(1-\nu^2)}A_{mn}^3(t) + f_{mn}\cos(\omega t). \quad (72)$$

Equation (72), for $c = 0$ and $f_{mn} = 0$, is identical to that obtained by Chu & Hermann (1956) for a rectangular plate when $L/b \to 0$, where b is the second dimension of the plate (infinity in present case). Chu & Herrmann's results were confirmed by several studies that followed (Dowell & Ventres 1968), so that successful comparison with it is particularly important. A similar result was obtained as a limiting case also by Dowell & Ventres (1968).

The second limiting case is $L/R \to \infty$; it corresponds to an infinitely long shell. If we multiply the equations by $R\lambda_m^2$, only the functions c_3, c_6 and c_{11} are different from zero in the particular solution F_p [equation (19)]. Considering the limiting case for $N_x = 0$, the homogeneous solution is simplified into $F_h = \frac{1}{2}x^2 Eh(n^2/8R^2)[A_{mn}^2(t) + B_{mn}^2(t)]$. The three equations of motion then are

$$\ddot{A}_{mn}(t)\rho h + \dot{A}_{mn}(t)ch + A_{mn}(t)D\lambda_m^4 = \frac{-Ehn^2}{R^3}A_{mn}(t)\left\{\frac{3}{16}[A_{mn}^2(t) + B_{mn}^2(t)]\frac{n^2}{R} - \frac{64}{15m\pi}[1-(-1)^m]A_{m0}(t)\right\} + f_{mn}\cos(\omega t), \quad (73)$$

$$\ddot{B}_{mn}(t)\rho h + \dot{B}_{mn}(t)ch + B_{mn}(t)D\lambda_m^4 = \frac{-Ehn^2}{R^3}B_{mn}(t)\left\{\frac{3}{16}[A_{mn}^2(t) + B_{mn}^2(t)]\frac{n^2}{R} - \frac{64}{15m\pi}[1-(-1)^m]A_{m0}(t)\right\}, \quad (74)$$

$$\ddot{A}_{m0}(t)10\rho h + \dot{A}_{m0}(t)10ch = \frac{-10Eh}{R^2}A_{m0}(t) + \frac{16}{15}\frac{Ehn^2}{m\pi R^3}[1-(-1)^m][A_{mn}^2(t) + B_{mn}^2(t)]. \quad (75)$$

These equations are similar to those obtained by Dowell (1967) for a ring, but the coefficients of the nonlinear terms are a little different. However, Dowell & Ventres (1968) observed that the ring solution is for zero axial bending and hence does not exactly correspond to the $L/R \to \infty$ limit of the cylindrical shell solution. Equations (73)–(75) can be reduced to those obtained by Evensen (1966) considering an inextensible ring, assuming that $F_h = 0$ and determining A_{m0} to satisfy continuity of the circumferential displacement.

8. NUMERICAL RESULTS AND DISCUSSION

Some numerical results, obtained with the aid of equations (34)–(36), were obtained for a few cases already studied in the literature, in order to allow a comparison. The first case analysed here was analytically and experimentally studied by Chen & Babcock (1975) and it relates to a circular cylindrical shell in vacuum, simply supported at the ends, having the following dimensions and properties: $L = 0{\cdot}2$ m, $R = 0{\cdot}1$ m, $h = 0{\cdot}247 \times 10^{-3}$ m, $E = 71{\cdot}02 \times 10^9$ Pa, $\rho = 2796$ kg/m^3 and $\nu = 0{\cdot}31$; the mode investigated is $n = 6$ and $m = 1$. In Figure 2 the undamped free vibration response calculated for amplitude $\tilde{A}_{mn} = h$ is shown; specifically, the functions $A_{mn}(t)$ and $A_{m0}(t)$ are plotted in Figure 2(a, b). The initial

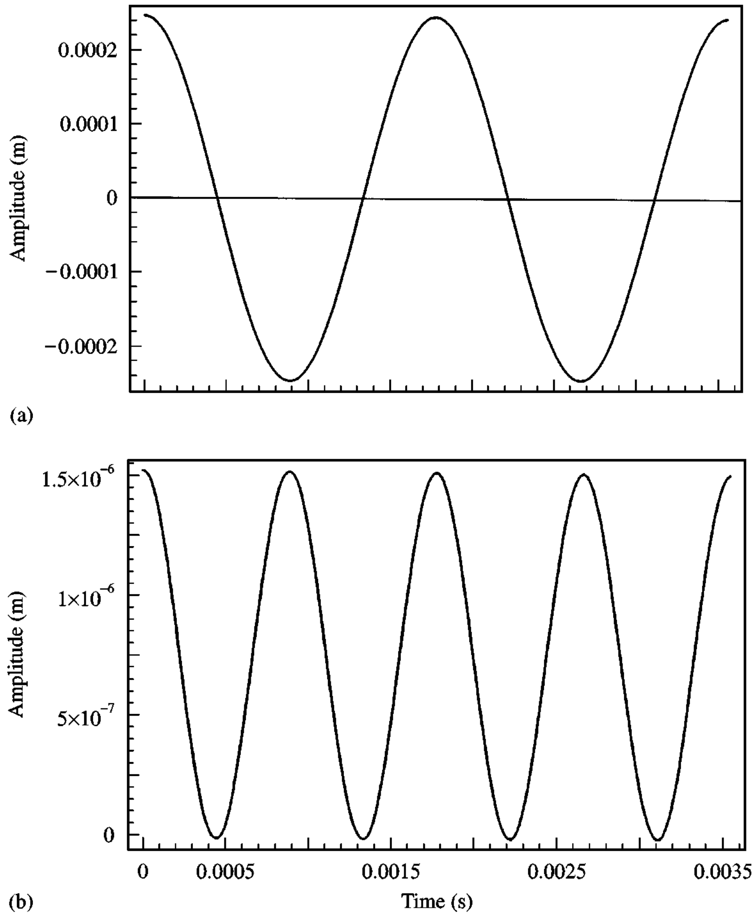

Figure 2. Free response calculated for vibration amplitude $\tilde{A}_{mn} = h$, for the shell studied by Chen & Babcock (1975): (a) $A_{mn}(t)$; (b) $A_{m0}(t)$.

condition $B_{mn}(t) = 0$ has been used, so that the companion mode is not activated. It is interesting to note that $A_{m0}(t)$ has twice the frequency of $A_{mn}(t)$ and that it is always positive; therefore $A_{m0}(t)$ effectively gives a shell contraction. This result is due to the nonlinear coupling and demonstrates that in this case the nonlinear normal mode corresponding to the linear mode $A_{mn}(t)$ is activated. This nonlinear normal mode oscillates at a frequency close to ω_{mn} and involves primarily the mode $A_{mn}(t)$ and less significantly the term $A_{m0}(t)$ which does not oscillate with the frequency of the linear axisymmetric mode $A_{m0}(t)$. The amplitude of $A_{m0}(t)$ is two orders of magnitude smaller than that of $A_{mn}(t)$.

For numerical integration of equations (34)–(36), the routine NDSolve of *Mathematica* software (Wolfram 1996) have been used. NDSolve switches between a non-stiff Adams method and a stiff Gear method through a Fehlberg order 4–5 Runge–Kutta method for non-stiff equations. Figure 2 was obtained by numerical integration, so that initial conditions must be accurately given in order to obtain the free response of the shell corresponding to a nonlinear normal mode. Several trials were made before finding the correct values.

To overcome this difficulty, all subsequent results were obtained by using the second-order perturbation solution. In particular, the frequency–amplitude relationship is

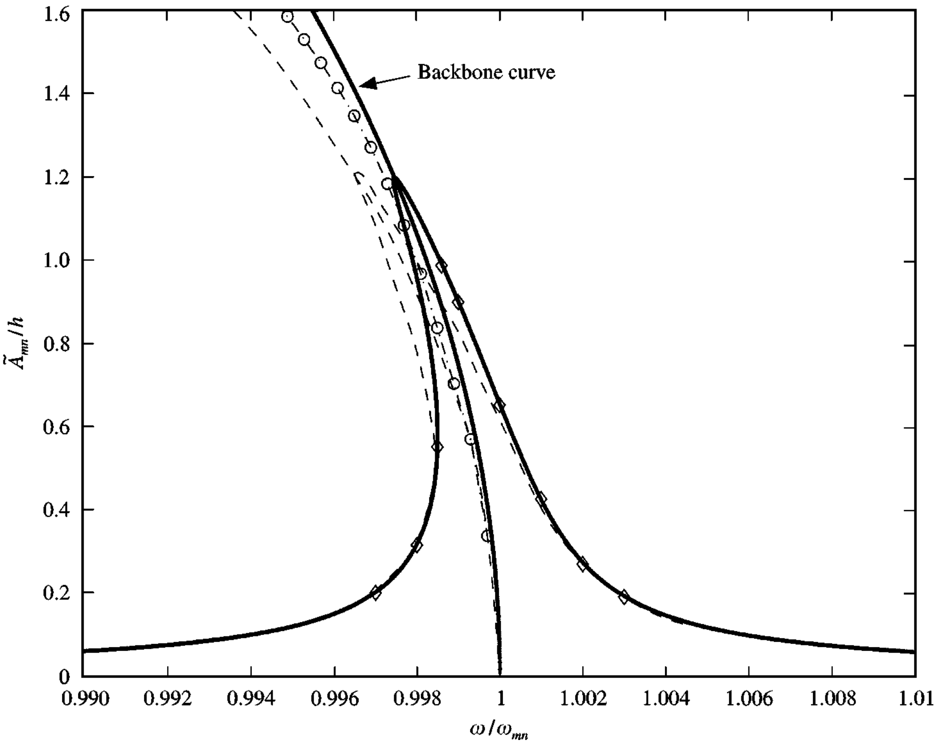

Figure 3. Response–frequency curves and backbone curves for the driven mode, for the shell studied by Chen & Babcock (1975). ——, Present perturbation results; ◇, present numerical integration results; – – –, Chen & Babcock (1975); –·–○–·–, backbone of Ganapathi & Varadan (1996).

computed (i) from equation (57) in the case of free undamped vibrations and gives the so-called backbone curve; (ii) from equation (62) in the case of forced vibrations with only the driven mode activated; (iii) from equation (63), by using the Newton–Raphson method, in the case of forced vibrations with companion mode participation.

In Figure 3 the response–frequency relationship of the driven mode is shown (continuous lines) and compared to that obtained by Chen & Babcock (1975) (dashed lines) for an amplitude of the external excitation $f_{mn} = 0{\cdot}0012h^2\rho\omega_{mn}^2$ and a damping ratio $2\zeta_{mn} = 0{\cdot}001$; the linear circular frequency is $\omega_{mn} = 2\pi \times 564{\cdot}2$ rad/s. The backbone curves (pertaining to undamped free vibrations) are also shown in the figure. Figure 3 shows good agreement between the present results and those obtained theoretically by Chen & Babcock (1975). The same case was also studied by Ganapathi & Varadan (1996), only for free vibrations; they obtained a backbone curve, shown in Figure 3, in very good agreement with the present results. The nonlinearity is of softening type in Figure 3. Results obtained by numerical integration of equations (34)–(36) by using the software *Mathematica* are given in the same figure; they are in very good agreement with results obtained by the perturbation solution.

The response–frequency relationship is shown in Figure 4 for the same case, but including the participation of the companion mode. In particular, Figure 4(a) shows the normalized amplitude $\tilde{A}_{mn}/h$ versus the frequency ratio ω/ω_{mn}, and Figure 4(b) shows the two normalized amplitudes, $\tilde{A}_{mn}/h$ and $\tilde{B}_{mn}/h$, in the case of companion mode participation. In Figure 4(a) both the solutions for $B_{mn}(t) = 0$ (driven mode only) and $B_{mn}(t) \neq 0$ (companion

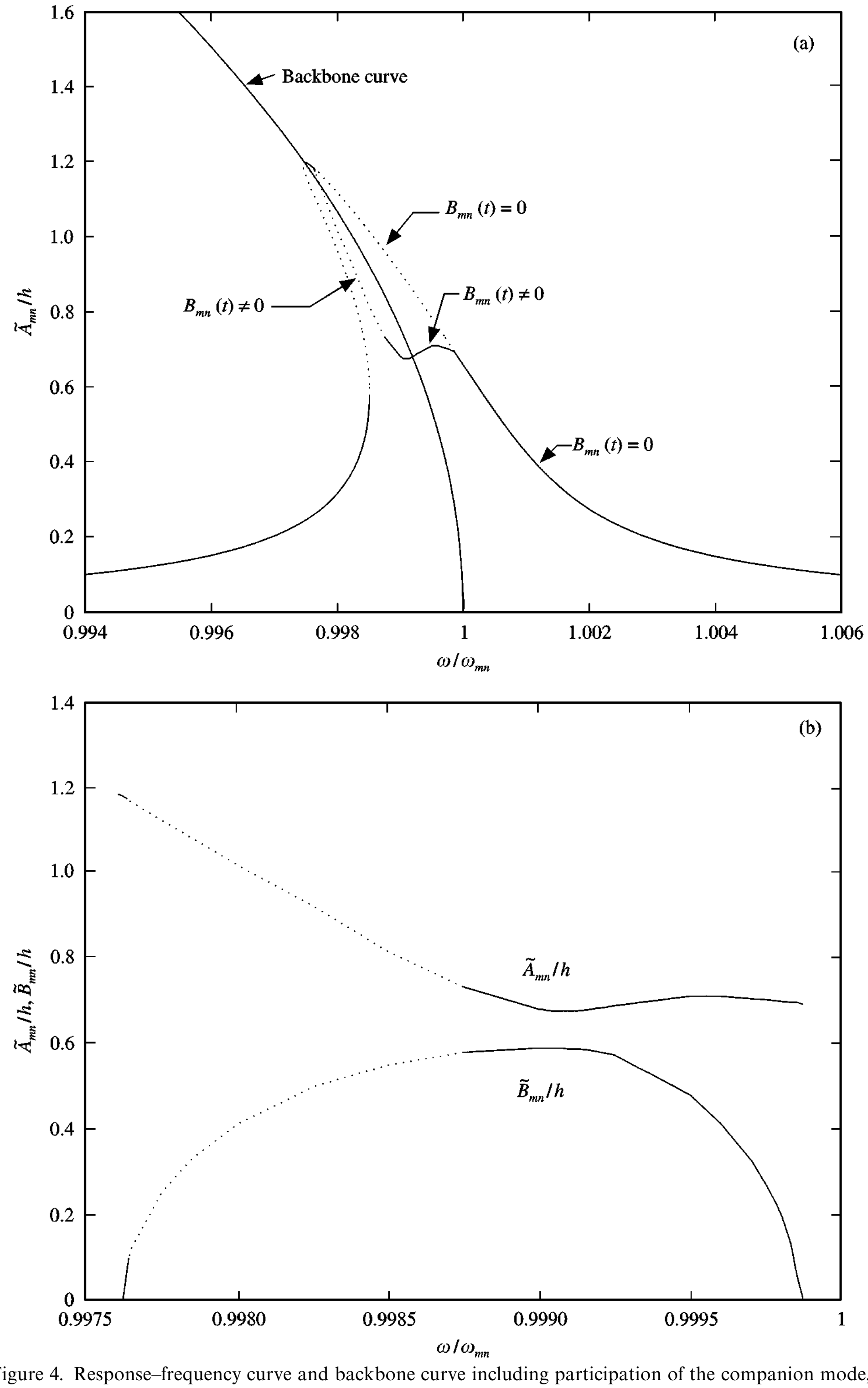

Figure 4. Response–frequency curve and backbone curve including participation of the companion mode, for the shell studied by Chen & Babcock (1975). ——, Stable solutions · · · ·, unstable solutions. (a) Driven mode, normalized amplitude $\tilde{A}_{mn}/h$; (b) Driven mode and companion mode when there is companion mode participation, normalized amplitudes $\tilde{A}_{mn}/h$ and $\tilde{B}_{mn}/h$.

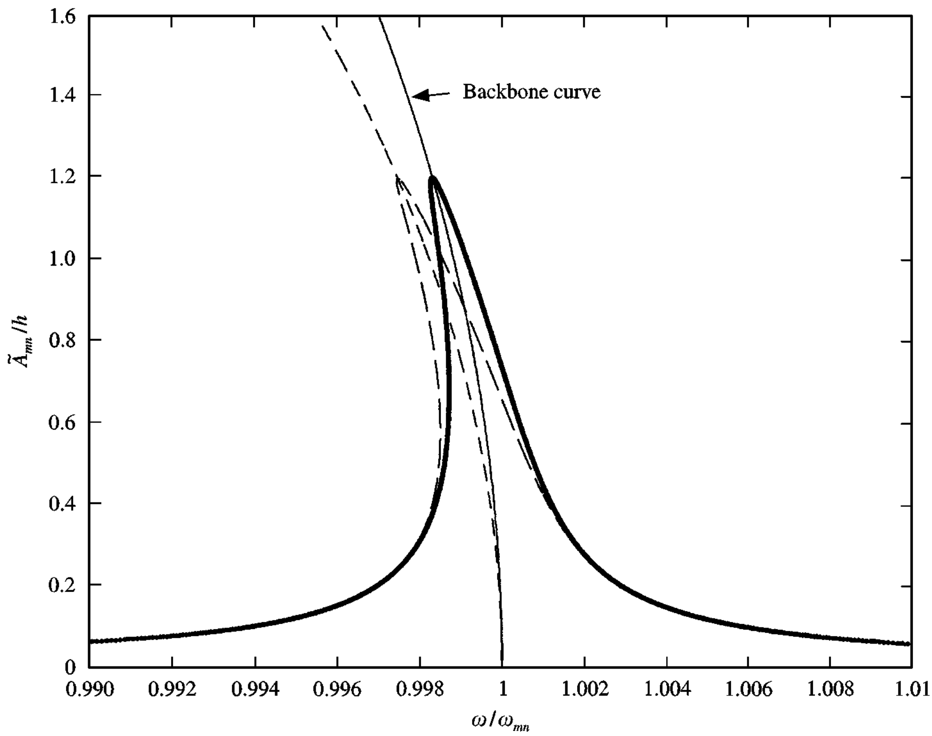

Figure 5. Response–frequency curves and backbone curves for the driven mode, for the shell studied by Chen & Babcock (1975), ——, Shell with axial displacement $u = 0$ at the shell ends; – – –, simply supported shell.

mode participation) are plotted together. In fact, both are solutions of equations (34)–(36) and give a multi-valued frequency–response function. The actual response in the multi-valued region must be determined by a stability analysis. Stable solutions are plotted as continuous lines and unstable solutions as dotted lines. Numerical results show that the extremities of the curve for $B_{mn}(t) \neq 0$ lie on the curve computed for $B_{mn}(t) = 0$. The present results are also in satisfactorily agreement with those of Chen & Babcock (1975), considering that they used a very different model. The response of the shell is affected by companion mode participation only in a narrow region around the backbone curve. Studying the response, we see that there exists a minimum amplitude $\tilde{A}_{mn}$ that must be exceeded in order to have participation of the companion mode in the case of damped vibrations; this minimum amplitude corresponds to a minimum amplitude f_{mn} of the external excitation. It is also interesting to observe that there is a region where no stable solutions arise. This peculiar behaviour confirms results obtained by Ginsberg (1973) in a very different way.

Figure 5 presents a comparison between the driven mode response of the simply supported shell (dashed lines), already presented in Figure 3, and the one obtained for zero axial displacement, $u = 0$, at the shell ends; see equation (9b) (continuous lines). We can see that the response relative to this second boundary condition, when axial displacement $u = 0$ at the ends, is slightly shifted to the right with respect to the previous case.

The second case analysed here was experimentally studied by Olson (1965) and theoretically by Evensen (1967) and Ganapathi & Varadan (1996). It deals with an empty circular cylindrical shell made of seamless copper, supported at the ends so as to have zero radial

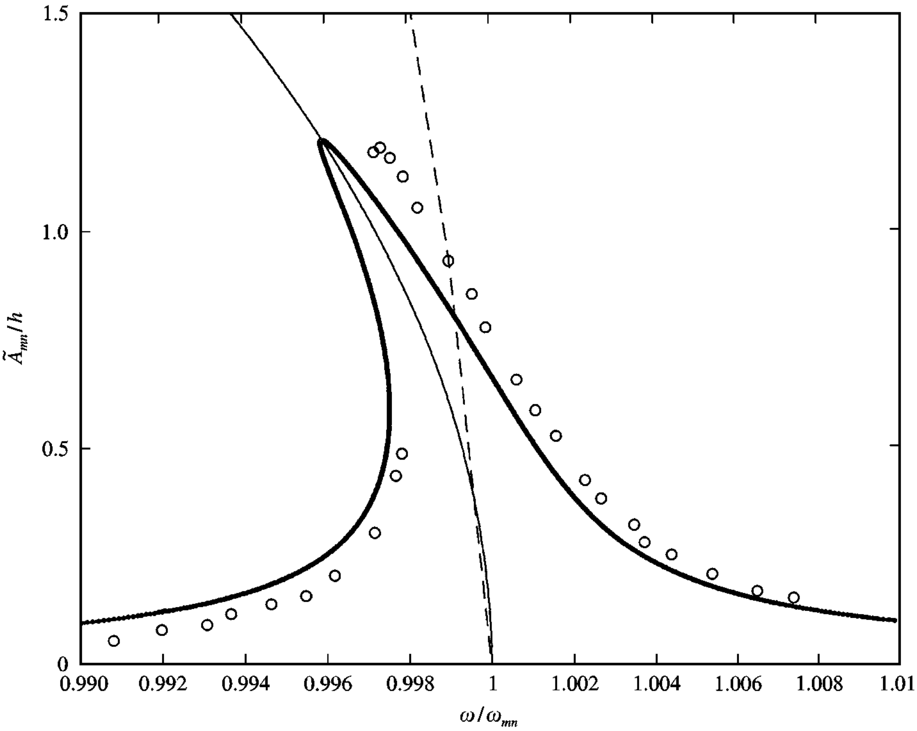

Figure 6. Response–frequency curves and backbone curves for the driven mode, for the shell studied by Olson (1965), ——, Present results; ○, Olson's experiments (1965) – – –, Evensen's backbone curve (1967).

deflection w, and for the following parameters: $L = 0{\cdot}39$ m, $R = 0{\cdot}203$ m, $h = 0{\cdot}11176 \times 10^{-3}$ m, $E = 124 \times 10^9$ Pa, $\rho = 8930$ kg/m^3 and $\nu = 0{\cdot}365$. The mode investigated is $n = 10$ and $m = 1$, and the amplitude of external excitation used in the simulations is $f_{mn} = 0{\cdot}002h^2\rho\omega_{mn}^2$ with a damping ratio $\zeta_{mn} = 0{\cdot}0008$; the linear circular frequency is $\omega_{mn} = 2\pi \times 91{\cdot}71$ rad/s. The present results for this case have been computed by imposing an axial displacement $u = 0$ at the ends. In fact, the constraint used by Olson (1965) in the experiments at the shell ends is closer to this case than to the classical simple support. In Figure 6 the simulated response–frequency relationship of the driven mode is shown (continuous line) and compared to experimental results obtained by Olson (1965) (circles). The backbone curve for this case (continuous line) is compared to that obtained by Evensen (1967) (dashed line); the backbone of Ganapathi & Varadan (1996) for this case is very close to that obtained by Evensen (1967) and is not shown in the figure. It may be concluded that, the present results are in good agreement with Olson's (1965) experiments.

The simply supported shell analytically studied by Gonçalves & Batista (1988) has been used for comparison in the case of a fluid-filled shell. Actually, the constraints to axial displacement used by Gonçalves & Batista only approximate a classical simple support; in fact, they imposed the condition $\partial u/\partial x = 0$ at the shell ends and $u = 0$ at $x = L/2$. The system studied has the following parameters: $L = 0{\cdot}5$ m, $R = 0{\cdot}83333$ m, $h = 4{\cdot}1666 \times 10^{-3}$ m, $E = 206 \times 10^9$ Pa, $\rho = 7576$ kg/m^3, $\rho_F = 1000$ kg/m^3 and $\nu = 0{\cdot}3$. The mode investigated is $n = 10$ and $m = 1$, and the amplitude of external excitation is $f_{mn} = 0{\cdot}003Eh^2/[R^2(1 - \nu^2)]$, without damping; the linear circular frequency is $\omega_{mn} = 2\pi \times 157$ rad/s. Comparison of the present response–frequency curve for the driven

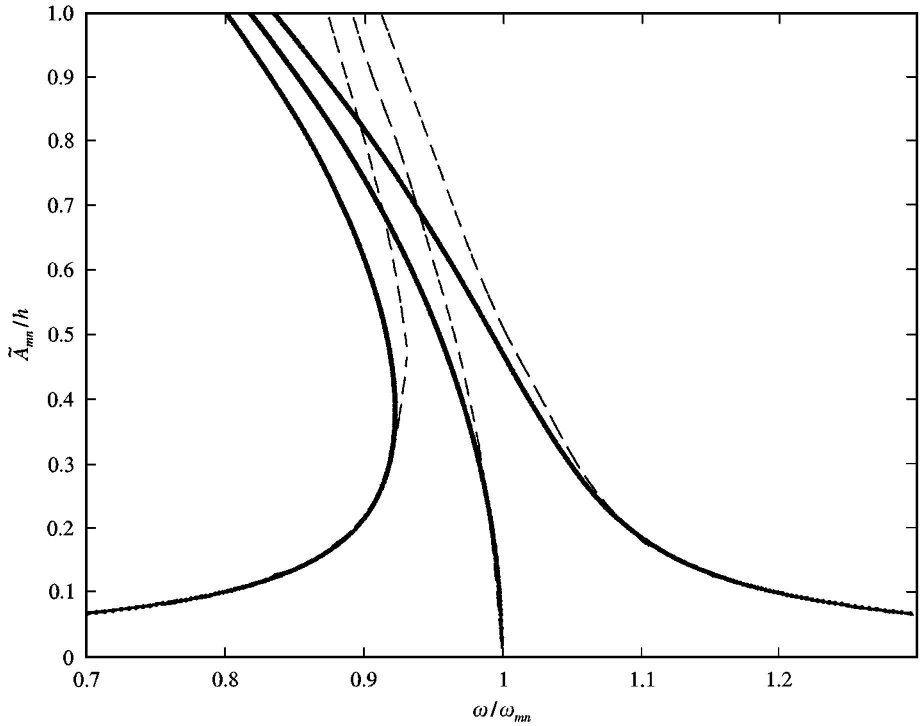

Figure 7. Response–frequency curves and backbone curves for the driven mode, for the water-filled shell studied by Gonçalves & Batista (1988). ——, Present results; ---, Gonçalves & Batista (1988).

mode only (continuous line) to that obtained by Gonçalves & Batista (1988) (dashed lines) is shown in Figure 7. Agreement is good, so long as $\tilde{A}_{mn}$ does not exceed 0·4h. Differences for larger vibration amplitudes are likely due to the fact that the present perturbation analysis was carried out only to second order, whereas Gonçalves & Batista's (1988) results were obtained numerically. This point will be investigated at the end of this section. The response–frequency relationship is shown in Figure 8 for the same case, but including the participation of companion mode and assuming a damping ratio $\zeta_{mn} = 0{\cdot}014$, which is reasonable for a water-filled shell (Amabili 1996). In this case, comparison is not possible because Gonçalves & Batista (1988) did not consider companion mode participation.

The effect of the damping ratio ζ_{mn} on shell response can be investigated by comparing Figures 8 and 9; Figure 9 was obtained for $\zeta_{mn} = 0{\cdot}010$. In particular, it is interesting to note the change in the shape of the response for both driven and companion modes, although the change in ζ_{mn} is modest. In particular, we see a more evident peak close to the right-hand bifurcation point.

Figure 10 shows the effects of the fluid inside and of the number of nodal diameters n on the response of the driven mode only. The effect of the fluid is to generate a stronger softening behaviour. The softening character of the response also increases with n, at least for the smaller amplitudes. It is interesting to note that, in this case, a significant reduction of frequencies arises for large amplitude vibrations, in contrast to what had been found for the cases studied by Chen & Babcock (1975) and Olson (1965) (and here in Figures 3 and 6). This fact is due to the different values of h/R and R/L in the three cases. In particular, $h/R = 0{\cdot}0025$ (Chen & Babcock), $h/R = 0{\cdot}00055$ (Olson), $h/R = 0{\cdot}005$

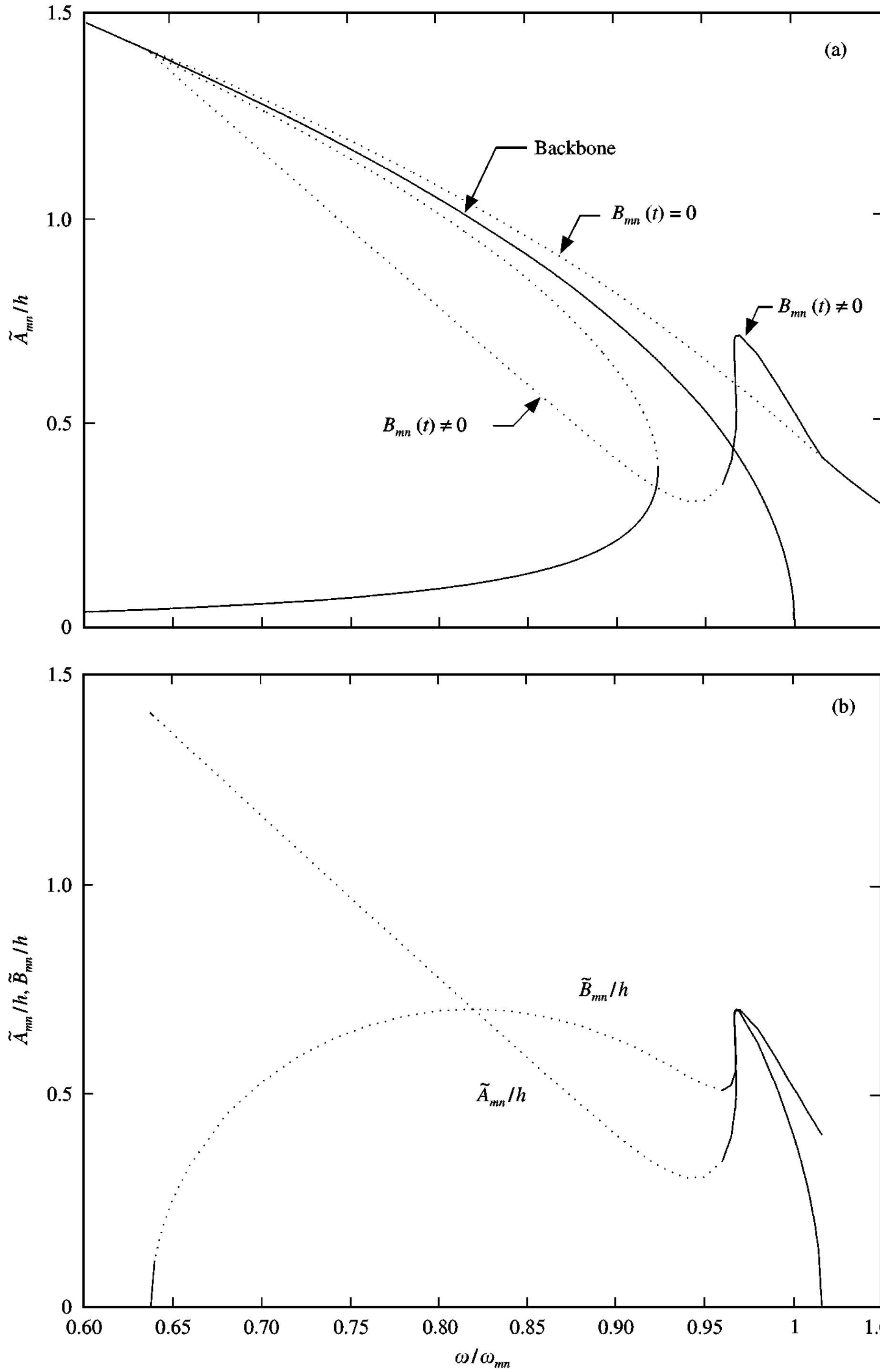

Figure 8. Response–frequency curves and backbone curves including the participation of companion mode, for the shell studied by Gonçalves & Batista (1988) with damping ratio $\zeta_{mn} = 0{\cdot}014$ (water-filled). ——, Stable solutions; · · · ·, unstable solutions. (a) Driven mode, normalized amplitude $\tilde{A}_{mn}/h$; (b) Driven mode and companion mode when there is companion mode participation, normalized amplitudes $\tilde{A}_{mn}/h$ and $\tilde{B}_{mn}/h$.

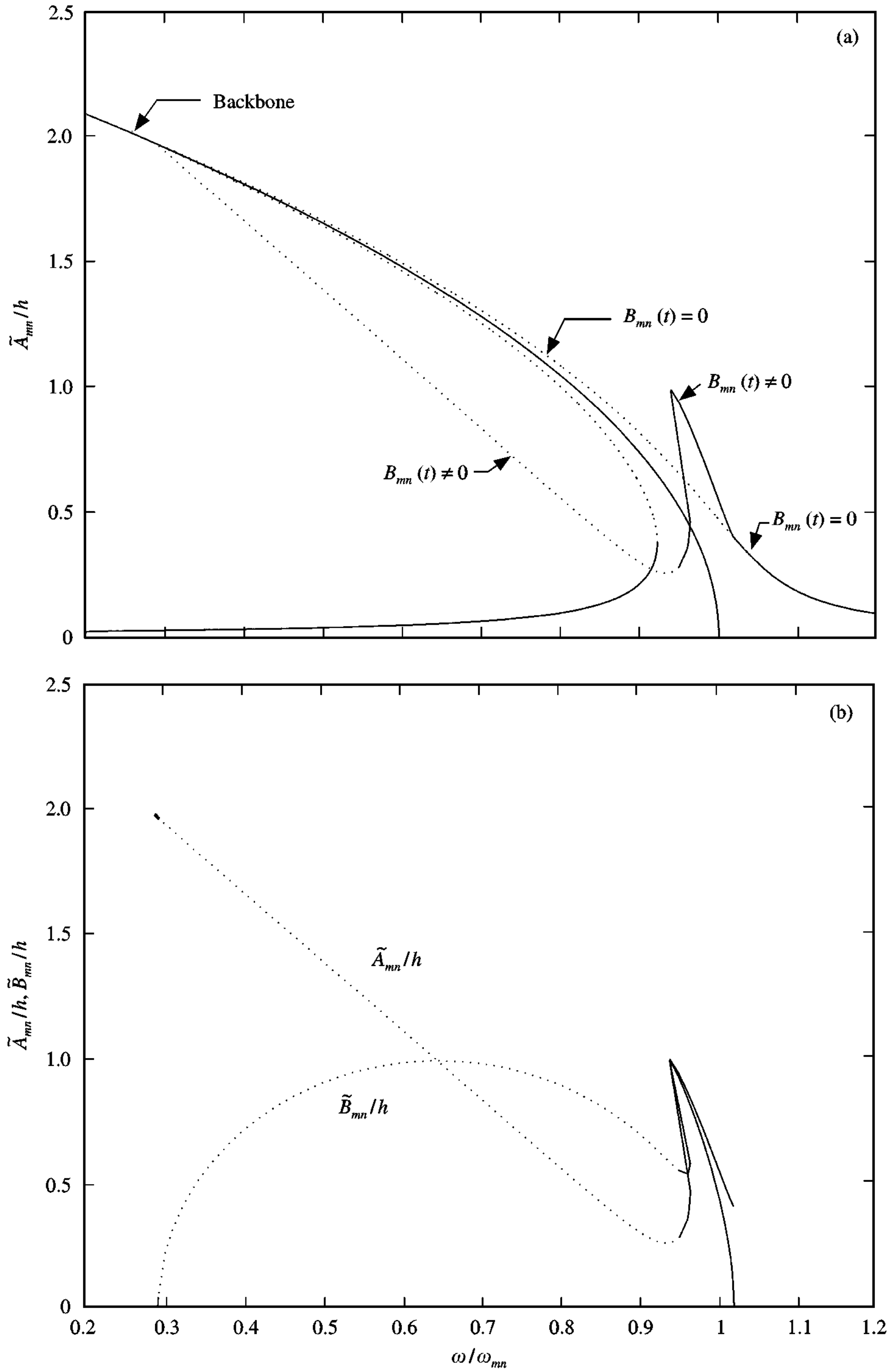

Figure 9. Response–frequency curves and backbone curves including the participation of companion mode, for the water-filled shell studied by Gonçalves & Batista (1988) with damping ratio $\zeta_{mn} = 0{\cdot}010$. ——, Stable solutions · · · ·, unstable solutions. (a) Driven mode, normalized amplitude $\tilde{A}_{mn}/h$; (b) Driven mode and companion mode when there is companion mode participation, normalized amplitudes $\tilde{A}_{mn}/h$ and $\tilde{B}_{mn}/h$.

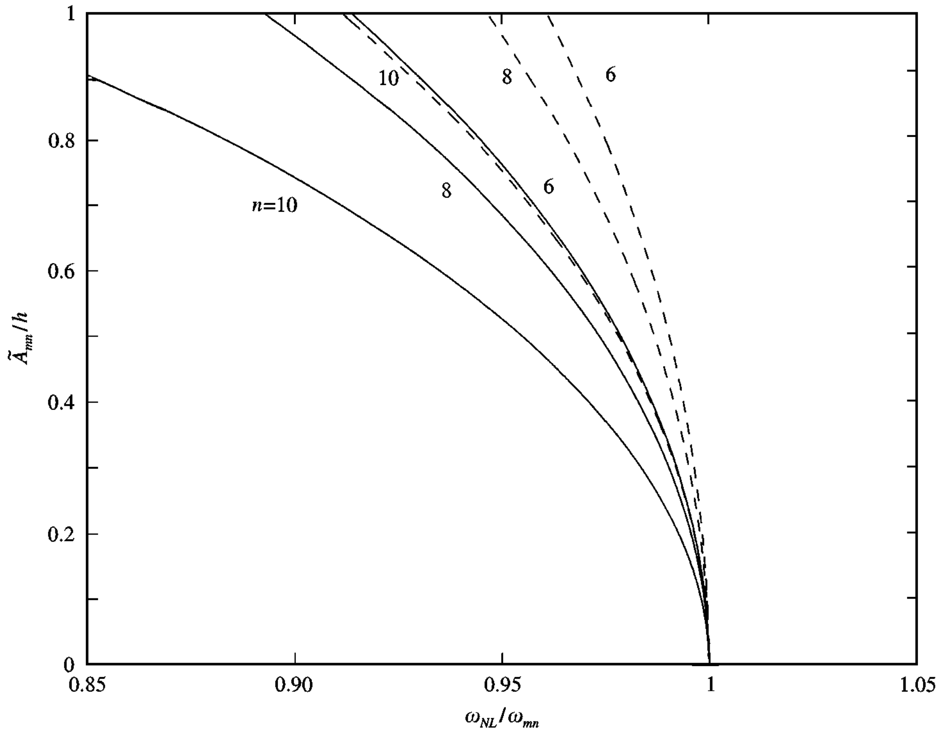

Figure 10. Backbone curves for the empty and water-filled shell for different n values, for the shell studied by Gonçalves & Batista (1988); ω_{NL} is the circular frequency of nonlinear vibrations. ——, Water-filled; – – –, empty shell.

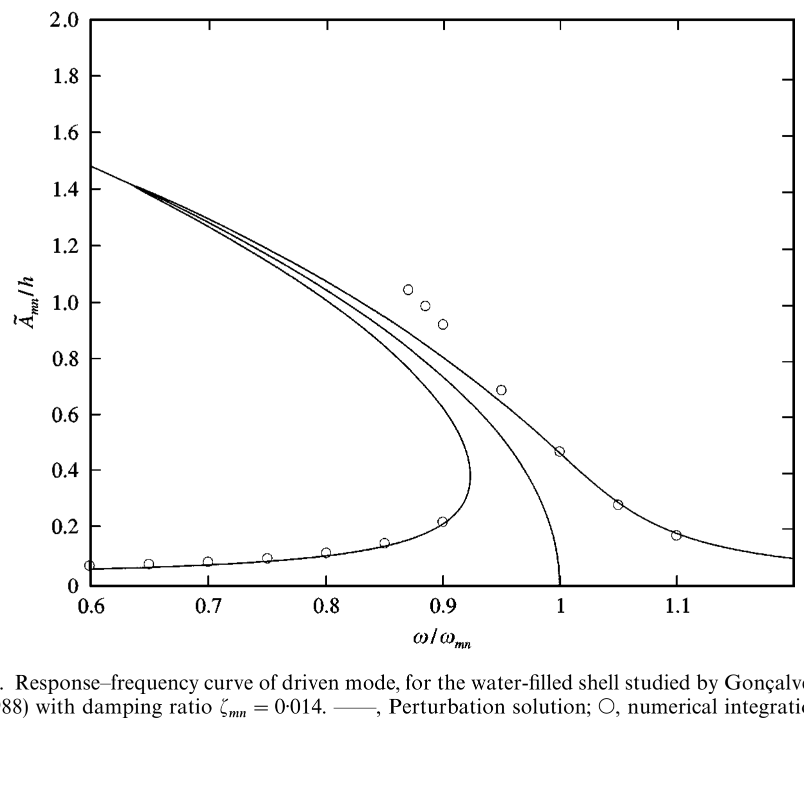

Figure 11. Response–frequency curve of driven mode, for the water-filled shell studied by Gonçalves & Batista (1988) with damping ratio $\zeta_{mn} = 0{\cdot}014$. ——, Perturbation solution; ○, numerical integration.

TABLE 1

Coefficients of equations (34)–(36) in the cases computed for comparison with Chen & Babcock's (1975), Olson's (1965) and Gonçalves & Batista's (1988) results. Note that k_3 and k_4 are different from zero only for boundary condition $u = 0$

Coefficients	Chen & Babcock	Olson	Gonçalves & Batista
ω_{mn}	3545	576·25	986·25
ζ	0·0005	0·0008	0
h_1	$-2{\cdot}520\times10^{12}$	$-4{\cdot}984\times10^{11}$	$-4{\cdot}284\times10^{9}$
h_2	$6{\cdot}182\times10^{13}$	$1{\cdot}714\times10^{13}$	$3{\cdot}225\times10^{10}$
h_3	$4{\cdot}834\times10^{13}$	$7{\cdot}008\times10^{12}$	$1{\cdot}689\times10^{11}$
$\tilde{f}_{mn}$	0·84897	0·074076	160·08
ω_{m0}	50401	19152	2527.6
k_1	$-6{\cdot}286\times10^{10}$	$-1{\cdot}246\times10^{10}$	$-5{\cdot}839\times10^{7}$
k_2	$2{\cdot}419\times10^{12}$	$3{\cdot}504\times10^{11}$	$4{\cdot}558\times10^{9}$
k_3	0	$-8{\cdot}567\times10^{9}$	0
k_4	0	$5{\cdot}464\times10^{11}$	0

(Gonçalves & Batista); $R/L = 0{\cdot}5$ (Chen & Babcock), $R/L = 0{\cdot}52$ (Olson), $R/L = 1{\cdot}66$ (Gonçalves & Batista). Thicker and shorter shells seem to be associated with more significant softening behaviour.

Figure 11 shows the frequency–response curve without companion mode participation, as obtained by perturbation analysis [already plotted in Figure 8(a)] and also numerical integration of equations (34)–(36) by using the software *Mathematica*, as already described. This figure is useful to show the degree of accuracy obtained by the perturbation solution. We can see that significant differences arise when $\tilde{A}_{mn}$ exceeds $0{\cdot}6h$. In fact, a significant reduction of natural frequency is observed in this case; the perturbation solution is very accurate only in the lower-amplitude region. Figure 11 clarifies the reason for the differences between the present results and Gonçalves & Batista's (1988) for larger amplitudes, as previously discussed.

The coefficients of equations (34)–(36) in the cases computed for comparison with Chen & Babcock's (1975), Olson's (1965) and Gonçalves & Batista's (1988) results are given in Table 1 for completeness.

9. CONCLUSIONS

The present results match well authoritative data available in the literature, confirming the accuracy of the proposed solution. This is particularly evident when comparing the nonlinear response curves. Moreover, it seems that the present study is the first one allowing companion mode participation in the presence of interaction between the shell and a dense fluid.

The effect of the fluid is to enhance the nonlinear character of shell vibration: indeed, a strongly softening behaviour is found for a fluid-filled shell. Hence, nonlinear analysis could be more important for fluid-filled shells than for empty ones.

The proposed method allows the study of many nonlinear aspects of the response, e.g. the backbone curve, response–frequency relationship, companion mode participation, travelling wave, different shell constraints, axisymmetric prestress, internal and external dense fluid, still retaining a relatively simple formulation. It is very important to note that the present approach exactly satisfies boundary conditions on radial displacement and continuity of circumferential displacement. Moreover, the use of Normal Forms in the perturbation analysis allows fast and reliable computations; in fact, numerical integration can be avoided and the problem of needing to determine appropriate initial conditions is skipped.

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APPENDIX A: TIME FUNCTIONS USED IN EQUATION (19)

The functions $c_i(t)$, $i = 1, \dots, 13$, used in equation (19) are given by

$$c_1(t) = -\frac{3EhR^2\lambda_m^2}{2n^2}A_{m0}(t)A_{mn}(t), \qquad c_2(t) = -\frac{3EhR^2\lambda_m^2}{2n^2}A_{m0}(t)B_{mn}(t), \tag{A1, A2}$$

$$c_3(t) = \frac{3Eh}{\lambda_m^2 R}A_{m0}(t), \qquad c_4(t) = \frac{Eh\lambda_m^2}{R\left(\lambda_m^2 + \dfrac{n^2}{R^2}\right)^2}A_{mn}(t), \tag{A3, A4}$$

$$c_5(t) = \frac{Eh\lambda_m^2}{R\left(\lambda_m^2 + \dfrac{n^2}{R^2}\right)^2}B_{mn}(t), \qquad c_6(t) = \frac{Ehn^2(A_{mn}^2(t) + B_{mn}^2(t))}{32\lambda_m^2 R^2}, \tag{A5, A6}$$

$$c_7(t) = \frac{EhR^2\lambda_m^2(-A_{mn}^2(t) + B_{mn}^2(t))}{32n^2}, \qquad c_8(t) = -\frac{EhR^2\lambda_m^2 A_{mn}(t)B_{mn}(t)}{16n^2}, \tag{A7, A8}$$

$$c_9(t) = \frac{12Ehn^2\lambda_m^2}{2R^2\left(4\lambda_m^2 + \dfrac{n^2}{R^2}\right)^2}A_{m0}(t)A_{mn}(t), \qquad c_{10}(t) = \frac{12Ehn^2\lambda_m^2}{2R^2\left(4\lambda_m^2 + \dfrac{n^2}{R^2}\right)^2}B_{m0}(t)B_{mn}(t), \tag{A9, A10}$$

$$c_{11}(t) = -\frac{Eh}{9\lambda_m^2 R}A_{m0}(t), \qquad c_{12}(t) = -\frac{9Ehn^2\lambda_m^2}{2R^2\left(16\lambda_m^2 + \dfrac{n^2}{R^2}\right)^2}A_{m0}(t)A_{mn}(t), \tag{A11, A12}$$

$$c_{13}(t) = -\frac{9Ehn^2\lambda_m^2}{2R^2\left(16\lambda_m^2 + \dfrac{n^2}{R^2}\right)^2}A_{m0}(t)B_{mn}(t). \tag{A13}$$

APPENDIX B: CONTINUITY OF CIRCUMFERENTIAL DISPLACEMENT

The continuity condition of the circumferential displacement v is satisfied on the average in equation (21). In this appendix the exact condition is applied in order to verify the accuracy of the approximation used in the main body of the paper. The exact continuity condition of the circumferential displacement v is

$$\int_0^{2\pi} \frac{\partial v}{\partial \theta}\,\mathrm{d}\theta = 0. \tag{B1}$$

For both of the two constraints at the shell ends studied, by using equations (3)–(5), equation (B1) is transformed into

$$\int_0^{2\pi} \left[\frac{1}{Eh}\left(\frac{\partial^2 F}{\partial x^2} - \nu \frac{\partial^2 F}{R^2 \partial \theta^2}\right) + \frac{w}{R} - \frac{1}{2}\left(\frac{\partial w}{R\partial\theta}\right)^2 \right] \mathrm{d}\theta = 0. \tag{B2}$$

After some manipulations, the following result is obtained:

$$\begin{aligned} &A_{m0}(t)\,[(-1)^m - 1]\,(8/(3m\pi R)) + (n^2/(8R^2))\,[A_{mn}^2(t) + B_{mn}^2(t)] + [1 - (-1)^m]\,\lambda_m [c_3(t) + 3c_{11}(t)] \\ &\quad /(EhL) - \lambda_m^2\,[c_3(t)\sin(\lambda_m x) + 4c_6(t)\cos(2\lambda_m x) + 9c_{11}(t)\sin(3\lambda_m x)]/(Eh) + (A_{m0}(t)/R) \\ &\quad \times [3\sin(\lambda_m x) - \sin(3\lambda_m x)] - (n^2/(4R^2))\,[A_{mn}^2(t) + B_{mn}^2(t)]\sin^2(\lambda_m x) = 0. \end{aligned} \tag{B3}$$

By using equations (A3), (A6), (A11) and a simple trigonometric transformation, it is easily verified that equation (B3) is identically verified for all x values. Therefore, it is the proof that the continuity condition is exactly satisfied by the present approach.

APPENDIX C: COEFFICIENTS OF NORMAL FORMS

The coefficients of the Normal Form given in equation (54) are:

$$\begin{aligned} g_{3,1}(\boldsymbol{\eta}) = {} & \mathrm{j}\,\frac{-\tilde{h}_1\tilde{k}_2 + \tilde{h}_2\omega_3^2}{\omega_1\omega_3^2}\,\eta_1\eta_2\eta_5 - \mathrm{j}\,\frac{\tilde{h}_1\tilde{k}_1 + 4\tilde{h}_2\omega_1^2 - \tilde{h}_2\omega_3^2}{-8\omega_1^3 + 2\omega_1\omega_3^2}\,\eta_2^2\eta_4 \\ & + \mathrm{j}\,\frac{8\tilde{h}_1\tilde{k}_1\omega_1^2 - 3\tilde{h}_1\tilde{k}_1\omega_3^2 + 12\tilde{h}_2\omega_1^2\omega_3^2 + 3\tilde{h}_2\omega_3^4}{-8\omega_1^3\omega_3^2 + 3\omega_1\omega_3^4}\,\eta_1^2\eta_4 \\ & + \mathrm{j}\left(\frac{\tilde{h}_3}{\omega_1} - \frac{\tilde{h}_1^2}{4\omega_1^2(2\omega_1 - \omega_3)} - \frac{\tilde{h}_1\tilde{k}_3}{\omega_1\omega_3^2} - \frac{\tilde{h}_1^2}{4\omega_1^2(2\omega_1 + \omega_3)}\right)\eta_1\eta_3\eta_6, \end{aligned} \tag{C1}$$

$$\begin{aligned} g_{3,2}(\boldsymbol{\eta}) = {} & \mathrm{j}\,\frac{-\tilde{h}_1\tilde{k}_1 + \tilde{h}_2\omega_3^2}{\omega_1\omega_3^2}\,\eta_1\eta_2\eta_4 - \mathrm{j}\,\frac{\tilde{h}_1\tilde{k}_1 + 4\tilde{h}_2\omega_1^2 - \tilde{h}_2\omega_3^2}{-8\omega_1^3 + 2\omega_1\omega_3^2}\,\eta_1^2\eta_5 \\ & + \mathrm{j}\,\frac{8\tilde{h}_1\tilde{k}_1\omega_1^2 - 3\tilde{h}_1\tilde{k}_1\omega_3^2 - 12\tilde{h}_2\omega_1^2\omega_3^2 + 3\tilde{h}_2\omega_3^4}{-8\omega_1^3\omega_3^2 + 3\omega_1\omega_3^4}\,\eta_2^2\eta_5 \\ & + \mathrm{j}\left(\frac{\tilde{h}_3}{\omega_1} - \frac{\tilde{h}_1^2}{4\omega_1^2(2\omega_1 - \omega_3)} - \frac{\tilde{h}_1\tilde{k}_3}{\omega_1\omega_3^2} - \frac{\tilde{h}_1^2}{4\omega_1^2(2\omega_1 + \omega_3)}\right)\eta_2\eta_3\eta_6, \end{aligned} \tag{C2}$$

$$\begin{aligned} g_{3,3}(\boldsymbol{\eta}) = {} & \mathrm{j}\,\frac{-10\tilde{k}_3^2 + 9\tilde{k}_4\omega_3^2}{\omega_3^3}\,\eta_3^2\eta_6 \\ & + \mathrm{j}\left(-\frac{2\tilde{k}_1\tilde{k}_3}{\omega_3^2} + \frac{\tilde{k}_2}{\omega_3} + \frac{\tilde{h}_1\tilde{k}_1}{-4\omega_1^2\omega_3 + 2\omega_1\omega_3^2} - \frac{\tilde{h}_1\tilde{k}_1}{4\omega_1^2\omega_3 + 2\omega_1\omega_3^2}\right)(\eta_1\eta_3\eta_4 + \eta_2\eta_3\eta_5). \end{aligned} \tag{C3}$$